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MAA

MATHEMATICS MAGAZINE

Integers

- Nonunique factorization
- Torricelli's take on areas and volumes
- Median triangles and others like them
- Triangles with trisectible angles

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LETTER FROM THE EDITOR

Some rings, like the integers, have the unique factorization property. In some other rings, unique factorization fails, but in ways that we can understand, so that we can still use the factorization properties to solve problems. Understanding how close a multiplicative structure comes to unique factorization is a recurring theme of ring theory. It turns out that some of these questions are better addressed in the presence of multiplication alone—that is, in the context of monoids, and not rings. That is the idea of our first article: Scott Chapman shows how some monoids have unique factorization, and some fail in large and small ways, all well worth studying.

Andrew Leahy's article is about Torricelli's work on the quadrature of the parabola, which was published in 1644. Torricelli started with the famous theorem of Archimedes, showing a way to calculate an area bounded by a parabola and a line. Leahy describes how Torricelli sought to unify many of Archimedes's results on areas and volumes—including "On the Sphere and the Cylinder" and his work on centers of gravity—and how Torricelli found in them a "common bond of truth." We now recognize that common bond as the integral calculus.

Triangles appear in Leahy's article, and reappear more prominently in the two articles that follow it. It is well known that the medians of any triangle can be translated to form another triangle, and that these two triangles are related in some attractive ways. Are there other constructions like this one? Yes, and some are quite special. All is revealed in the article on "outer median triangles" by Árpad Bényi and Branko Ćurgus.

As we know, some angles (like 45°) can be trisected using straightedge and compass, and some angles (like 60°) cannot be. (Of course, better tools would help; see page 228.) That puts the (classically) trisectible angles in a privileged class. Can a triangle be formed whose angles are all trisectible? More of a challenge: Can an *integer-sided* triangle be formed whose angles are all trisectible? Now that is a question worthy of attack, and Russ Gordon guides us in the adventure, using powerful (but always "elementary") methods of number theory.

We have a very special compilation by Felix Lazebnik. Mathematics is full of surprises, and here are 23 of his favorites. You'll find some of them familiar, and many of them thought-provoking. For the last four items, the biggest surprises may still be to come; please send your solutions to these problems to this MAGAZINE.

In the Notes Section B. W. Corson presents a trisection tool, David Brink provides a proof of "$\pi^2/6$" citing Euclid, and Vincent Coll and Maria Qirjollari extend a result about volumes of revolution to higher dimensions.

We also have a Crossword Puzzle by Brendan Sullivan (page 196). There will be another in December. Perhaps these will become a regular feature. They will surely appear in some form, and we will look forward to them.

Walter Stromquist, Editor

ARTICLES

A Tale of Two Monoids: A Friendly Introduction to Nonunique Factorizations

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It was the best of times, it was the worst of times...

The notion of unique factorization is central to almost every basic course in number theory and abstract algebra. It is an essential property of the integers, but it is not shared by all integer-like systems. Over the last thirty years, a vast literature has appeared that discusses arithmetical systems in which unique factorization fails. An extensive bibliography is given by Geroldinger and Halter-Koch [**9**]. While much of this literature is inaccessible to undergraduates, many of the ideas are relatively simple. In this paper, we present some of these ideas using two basic examples.

By a "friendly" introduction we mean that our techniques are taken from a first course in number theory. Let $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ be the integers, $\mathbb{N} = \{1, 2, 3, \ldots\}$ the natural numbers, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the natural numbers together with zero, and $\mathbb{Q}$ the rationals. The congruence relations on $\mathbb{Z}$ are defined, for each $n \in \mathbb{N}$, by

$$a \equiv b \pmod{n} \qquad \text{if and only if} \qquad n \mid a - b.$$

Among the nice properties of congruence is that, if $a \equiv b \pmod{n}$ and $c \in \mathbb{Z}$, then $ca \equiv cb \pmod{n}$. But the converse of this property does not always hold: $ca \equiv cb \pmod{n}$ does not imply that $a \equiv b \pmod{n}$.

The monoids

We consider two sequences:

$$1,\ 5,\ 9,\ 13,\ 17,\ 21,\ 25,\ 29,\ 33,\ 37,\ 41,\ 45,\ 49,\ \ldots \tag{1}$$

and

$$1,\ 4,\ 10,\ 16,\ 22,\ 28,\ 34,\ 40,\ 46,\ 52,\ 58,\ 64,\ 70,\ 76,\ 82,\ \ldots. \tag{2}$$

Except for the first term in (2), they are both arithmetic progressions. We can write the first sequence as

$$\mathbf{H} = \{1 + 4k \mid k \in \mathbb{N}_0\} = 1 + 4\mathbb{N}_0$$

and the second sequence as

$$\mathbf{M} = \{1\} \cup \{4 + 6k \mid k \in \mathbb{N}_0\} = \{1\} \cup (4 + 6\mathbb{N}_0).$$

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While these two sets can be defined recursively by addition, they share a stronger property. For instance, if $1 + 4k$ and $1 + 4t$ are in **H**, then

$$(1 + 4k)(1 + 4t) = 1 + 4(k + t + 4kt)$$

is also in **H**, and hence **H** is closed under multiplication. A similar argument (treating 1 as a special case) shows that **M** is also closed under multiplication.

Subsets of $\mathbb{N}$ that are, like **H** and **M**, multiplicatively closed and contain 1, are known as *monoids*. (In general, a monoid is any set with an associative operation and a unit element. We will be concerned especially with monoids that, like **H** and **M**, are subsets of $\mathbb{N}$.)

The monoid **H** is known as the *Hilbert monoid*. David Hilbert used this set in his elementary number theory courses to convince students of the necessity of proving the unique factorization property of the integers [**7**, 19–22] [**10**, Chapter 3.3]. To see Hilbert's point, in $\mathbf{H} = 1 + 4\mathbb{N}_0$ we have

$$693 = 21 \cdot 33 = 9 \cdot 77.$$

By viewing the prime factorizations of these numbers in $\mathbb{Z}$, namely $9 = 3 \cdot 3$, $21 = 3 \cdot 7$, $33 = 3 \cdot 11$, and $77 = 7 \cdot 11$, it is clear that none of them can be factored further in $1 + 4\mathbb{N}_0$. Thus, 693 does not factor uniquely in **H** into factors that do not further nontrivially decompose. (While 9, 21, 33, and 77 cannot be factored in **H**, we hesitate to call them "primes." We will see that their multiplicative behavior in **H** is much different from that of prime numbers in $\mathbb{Z}$.)

Similarly, in **M** we have

$$1540 = 70 \cdot 22 = 154 \cdot 10,$$

and clearly $70 = 2 \cdot 5 \cdot 7$, $22 = 2 \cdot 11$, $154 = 2 \cdot 7 \cdot 11$, and $10 = 2 \cdot 5$ cannot be factored further in $4 + 6\mathbb{N}_0$. We will call **M** *Meyerson's monoid* (see [**3**] and [**4**] for in-depth studies of this monoid).

While both **H** and **M** contain instances of nonunique factorizations, the nonunique factorizations in **H** behave especially well, while those in **M** do not. Much of the rest of the paper will be dedicated to demonstrating this contrast. We will see that, in some sense, factoring in **H** will be the "best of times" and factoring in **M** the "worst of times."

Other examples Both **H** and **M** are extremely easy to define. Their simplicity makes them especially useful as classroom examples of systems without unique factorization.

The first example of nonunique factorization presented in many abstract algebra courses occurs in the integral domain $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$. Like **H** and **M**, $\mathbb{Z}[\sqrt{-5}]$ is a multiplicative monoid. In $\mathbb{Z}[\sqrt{-5}]$,

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}),$$

demonstrating that 6 factors in two different ways into products of irreducible elements of $\mathbb{Z}[\sqrt{-5}]$. But to verify this, we must introduce the notion of units (defined below) and show that 2 is not a unit multiple of either $1 + \sqrt{-5}$ or $1 - \sqrt{-5}$. Another popular example occurs in the integral domain $\mathbb{Z}[2i]$, where we have $-4 = (-2)(2) = (2i)(2i)$; but again, we must deal with units and complex numbers.

Arithmetical congruence monoids Both **H** and **M** are examples of a more general class of multiplicative monoids. Let a and b be natural numbers with $a \leq b$. If $a \equiv a^2$

(mod b), then

$$(a+bk)(a+bt) = a^2 + b(ak + at + bkt) \equiv a^2 \equiv a \pmod{b},$$

and hence the set

$$M(a,b) = \{a + bk \mid k \in \mathbb{N}_0\} \cup \{1\}$$

is closed under multiplication, and is a multiplicative monoid. A monoid of the form $M(a,b)$ for natural numbers $a \leq b$ is called an *arithmetical congruence monoid* or *ACM*. (See [**3**] and [**4**] for more about these algebraic objects.)

The basics of factorization theory

As noted above, a *monoid* M is a set with an associative operation and a *unit element*. We write the operation multiplicatively, so we call the unit element 1, and write its defining property as $1x = x1 = x$ for all $x \in M$. The monoid is called *commutative* if $xy = yx$ for all $x, y \in M$, and *cancellative* if whenever $ab = ac$ for a, b, and c in M, then $b = c$. Both **H** and **M** are examples of commutative cancellative monoids. In this paper, we consider only commutative, cancellative monoids.

We use the usual notation for divisibility: We say that $x \mid y$ in M if $xz = y$ for some $z \in M$. An element $u \in M$ is called a *unit* in M (not the same as a unit element) if $u \mid 1$, and equivalently, if u has an inverse $v \in M$ such that $uv = 1$. By M^*, we mean the set of elements of M that are not units.

If $x \mid y$ and $y \mid x$ in M, then we say that x and y are *associates*. In a cancellative monoid M, x and y are associates if and only if $x = uy$ where u is a unit of M.

If $x \in M^*$, then

- x is *prime* if whenever $x \mid yz$ for $x, y, z \in M$, then either $x \mid y$ or $x \mid z$; and
- x is *irreducible* (or an *atom*) if whenever $x = yz$ for $x, y, z \in M$, then y or z is a unit in M.

All primes are atoms (in a cancellative monoid), but (in some monoids) not all atoms are prime. To see that all primes are irreducible, suppose that x is a prime in M. If $x = yz$, then $x \mid yz$ and either $x \mid y$ or $x \mid z$. If $x \mid y$, then $x = yz = xwz$ for some w in M and, by cancellation, $wz = 1$. Hence, z is a unit of M. Similarly, if $x \mid z$, then y is a unit. Thus, x is irreducible in M. An example in **H** of an atom that is not prime is 9. We have seen that 9 is irreducible in **H**. Now $9 \mid 21 \cdot 33$ but $9 \nmid 21$ and $9 \nmid 33$, so 9 is not prime in **H**.

When primes are not the same as atoms, it turns out that we are mainly interested in factorization into atoms (and not factorization into primes). In some monoids such as **H** and **M**, we will see that some elements cannot be factored into primes at all, but every non-unit can be factored into atoms.

We let $\mathcal{A}(M)$ be the set of all atoms of M; that is, we define

$$\mathcal{A}(M) = \{x \mid x \text{ is irreducible in } M\}.$$

If every $x \in M^*$ can be written as a product of elements from $\mathcal{A}(M)$, then M is called *atomic*. That is, in an atomic monoid, every $x \in M^*$ can be written as

$$x = \alpha_1 \cdots \alpha_k$$

with each $\alpha_i \in \mathcal{A}(M)$. In this case, k is called the *length* of the factorization. Using a proof identical to that used in $\mathbb{Z}$ to show that every integer greater than 1 can be written as a product of prime integers (see [**5**]), we can see that both **H** and **M** are atomic monoids.

We call M a *unique factorization monoid* or a *UFM*, if, given any element $x \in M^*$, the following two conditions hold:

(a) there exist atoms $\alpha_1, \ldots, \alpha_k$ of M such that $x = \alpha_1 \cdots \alpha_k$,

(b) if $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_j$ are atoms of M with $x = \alpha_1 \cdots \alpha_k = \beta_1 \cdots \beta_j$, then $k = j$ and there is a permutation σ of $\{1, 2, \ldots, k\}$ such that α_i and $\beta_{\sigma(i)}$ are associates for each $1 \leq i \leq k$.

(This definition is simpler in the case of **H** and **M**; since 1 is the only unit in either monoid, elements are associates exactly when they are equal. So condition (2) means that the two factorizations are the same except for the order in which the atoms appear.)

EXAMPLE. Not all multiplicative monoids are atomic. Let $\mathbb{Z} + X\mathbb{Q}[X]$ be the multiplicative monoid of polynomials from $\mathbb{Q}[X]$ whose constant terms are integers. It is clear that the constant polynomials that correspond to the prime numbers in $\mathbb{Z}$ are irreducible in $\mathbb{Z} + X\mathbb{Q}[X]$. To illustrate that $\mathbb{Z} + X\mathbb{Q}[X]$ is not atomic, we argue that the polynomial $X \in \mathbb{Z} + X\mathbb{Q}[X]$ cannot be factored as a product of atoms. Clearly, X is not an atom, as $X = p(\frac{1}{p}X)$, where p is any positive prime, is a factorization of X into two nonunits. The same argument shows that qX is not an atom, where q is any nonzero element of $\mathbb{Q}$. Suppose that $X = f_1(X) \cdots f_k(X)$, where each $f_i(X)$ is an atom of $\mathbb{Z} + X\mathbb{Q}[X]$ and $k \geq 2$. By a degree argument, exactly one $f_i(X)$ is of degree one; the remainder are of degree 0. Without loss of generality, assume that $f_1(X) = qX + r$, where $q \in \mathbb{Q}$ and $r \in \mathbb{Z}$, has degree 1. Then $f_2(X) = z_2, \ldots, f_k(X) = z_k$ for nonzero integers $z_2, \ldots z_k$. It follows that $rz_2 \cdots z_k = 0$ and hence $r = 0$. Thus $f_1(X) = qX$ is an atom, which contradicts our previous observation that it is not. Hence $\mathbb{Z} + X\mathbb{Q}[X]$ is not atomic.

In the rest of this paper, we consider only commutative, cancellative, atomic monoids.

Each prime element of a commutative cancellative monoid M has a very important property that comes into play when the elements of M are written as products of irreducibles.

PROPOSITION 1. *Let M be an atomic commutative cancellative monoid. Suppose that y is a prime element of M and x any nonunit of M. If $y \mid x$, then some associate of y appears in every irreducible factorization of x in M. Thus, if M has a unique unit, then y appears in every irreducible factorization of x in M.*

Proof. Suppose $x = \alpha_1 \cdots \alpha_t$ is a factorization of x into irreducibles in M. Then $y \mid \alpha_1 \cdots \alpha_t$ in M, and by repeatedly using the fact that y is prime, we have that $y \mid \alpha_i$ for some i. Thus, $yz = \alpha_i$ for some z in M. Since α_i is irreducible, z is a unit and y and α_i are associates. The second statement now easily follows. ■

Since each of **H** and **M** has a unique unit, in both monoids any prime divisor y of an element x appears in every irreducible factorization of x.

Atoms and atomic factorizations in **H**

Which elements in **H** and **M** are irreducible? This is a key question if we are to better understand irreducible factorizations in these monoids.

The atoms of **H** can be nicely described in terms of the primes in $\mathbb{Z}$.

PROPOSITION 2. *An element x is irreducible in* **H** *if and only if either*

(I) *$x = p$, where p is a prime in $\mathbb{Z}$ and p is congruent to 1 modulu 4, or*

(II) *$x = p_1 p_2$, where p_1 and p_2 are primes in $\mathbb{Z}$ congruent to 3 modulo 4.*

The irreducibles of Type I are prime in **H***, while the irreducibles of Type II are not.*

Proof. ($\Rightarrow$) If x is irreducible in **H** and prime in $\mathbb{Z}$, then by definition $x \equiv 1 \pmod 4$. So assume that x is irreducible in **H** but not prime in $\mathbb{Z}$. Write $x = p_1 \cdots p_t$ for not-necessarily-distinct primes $p_1, \ldots, p_t$ in $\mathbb{Z}$ and $t \geq 2$. All elements of **H** are odd, so each prime p_i is congruent to 1 or 3 modulo 4. Suppose that one of the p_i—without loss of generality, we may assume it is p_1—is congruent to 1 modulo 4. Then

$$p_1 \cdot p_2 \cdots p_t \equiv 1 \pmod 4 \text{ implies that } p_2 \cdots p_t \equiv 1 \pmod 4, \tag{3}$$

which puts $p_2 \cdots p_t$ in **H**. Thus, x factors in **H** as $x = (p_1)(p_2 \cdots p_t)$, contradicting its irreducibility. Thus, each $p_i \equiv 3 \pmod 4$. Now suppose that $t > 2$. Since $p_1 p_2 \equiv 1 \pmod 4$, we have that

$$p_1 p_2 \cdot p_3 \cdots p_t \equiv 1 \pmod 4 \text{ implies that } p_3 \cdots p_t \equiv 1 \pmod 4, \tag{4}$$

which again yields a factorization of x as $(p_1 p_2)(p_3 \cdots p_t)$ in **H**. Thus $t = 2$, and x has the form specified for Type II.

($\Leftarrow$) If $x = p$ is of Type I, then a proper factorization of x in **H** would result in a proper factorization of x in $\mathbb{Z}$, contradicting the fact that x is prime in $\mathbb{Z}$. If $x = p_1 p_2$ is of Type II, then by unique factorization in $\mathbb{Z}$, the only possible factorization in **H** would be $x = (p_1)(p_2)$, but neither p_1 nor p_2 is in **H**. Hence x again is irreducible in **H**.

To verify the last statement of the theorem, if x is an irreducible of Type (I) and $x \mid yz$ in **H**, then in $\mathbb{Z}$ it must divide either y or z. Using the argument in (3) on either y or z yields that x divides either y or z in **H**. Thus, x is prime in **H**. Now, suppose $x = p_1 p_2$ is an irreducible of Type II. Pick primes p_3 and p_4, each congruent to 3 modulo 4 and each distinct from both p_1 and p_2. Set $y = p_3 p_4$, $z = p_1 p_3$, and $w = p_2 p_4$; note that y, z, and $w \in$ **H**. Then $xy = zw \in$ **H**, so since $y \in$ **H**, we have $x \mid zw$ in **H**. But $x \nmid z$ and $x \nmid w$ in **H**, so x is not prime in **H**. ∎

The proof of Proposition 2 allows us to demonstrate a nice property concerning nonunique factorizations in **H**.

COROLLARY. *Let $x \in$* **H***. If*

$$x = \alpha_1 \cdots \alpha_s = \beta_1 \cdots \beta_t$$

*for α_i and β_j in $\mathcal{A}$(***H***), then $s = t$.*

Proof. Factor x as $p_1 \cdots p_r$ with each p_i prime in $\mathbb{Z}$. Let c_α represent the number of irreducibles of Type I and d_α the number of irreducibles of Type II in the factorization $\alpha_1 \cdots \alpha_s$. In a similar manner, define c_β and d_β for the factorization $\beta_1 \cdots \beta_t$. By Proposition 1, $c_\alpha = c_\beta$. Then we must have $d_\alpha = \frac{1}{2}(r - c_\alpha) = \frac{1}{2}(r - c_\beta) = d_\beta$. ∎

The proof of the Corollary shows that irreducible factorizations of a element x in **H** are formed by "pairing up" the prime factors of x in $\mathbb{Z}$ to form Type II irreducible factors in **H**.

EXAMPLE. To help demonstrate the preceding corollary, let's produce all irreducible factorizations in **H** of

$$141{,}851{,}281 = 4 \times 35{,}462{,}820 + 1.$$

Now,

$$141{,}851{,}281 = 11 \times 13 \times 17 \times 23 \times 43 \times 59,$$

and rearranging the primes on the right-hand side, we obtain

$$\begin{aligned} 141{,}851{,}281 &= \underbrace{13 \times 17}_{p\equiv 1 (\text{mod } 4)} \times \underbrace{11 \times 23 \times 43 \times 59}_{p\equiv 3 (\text{mod } 4)} \\ &= 13 \times 17 \times (11 \times 23) \times (43 \times 59) = 13 \times 17 \times 253 \times 2537 \\ &= 13 \times 17 \times (11 \times 43) \times (23 \times 59) = 13 \times 17 \times 473 \times 1357 \\ &= 13 \times 17 \times (11 \times 59) \times (23 \times 43) = 13 \times 17 \times 649 \times 989. \end{aligned}$$

A monoid M with the property that for any $x \in M$, all factorizations of x into atoms have the same length, is called *half-factorial*. Thus, the assertion of the Corollary is that **H** is half-factorial. There is a rich recent history of the study of half-factorial monoids and integral domains, and the interested reader can find many interesting examples of such structures in [**11**]. Much of this history was motivated by the observation of Carlitz in [**6**] that the previously mentioned monoid $\mathbb{Z}[\sqrt{-5}]$ is half-factorial. His argument uses the fact that the *class number* of $\mathbb{Z}[\sqrt{-5}]$ is 2 (see [**12**] for a definition of class numbers).

Atoms and atomic factorizations in M

Now, since

$$10^4 \text{ and } 250 \times 10 \times 4$$

are both decompositions of 10,000 into products of irreducible elements of **M**, it follows that Meyerson's monoid is *not* half-factorial. But what is causing the factorizations to have different lengths? Consider a second example,

$$154^3 = 1372 \times 2662,$$

which decomposes into primes in $\mathbb{Z}$ as

$$(2 \times 11 \times 7)^3 = (4 \times 7^3) \times (2 \times 11^3).$$

The interplay between atoms of **M** exactly divisible by 2 (i.e., 2 divides the atom in $\mathbb{Z}$ but 2^2 does not) and atoms of **M** exactly divisible by $2^2 = 4$ is the key to answering the above question. To determine the atoms of **M**, we begin with a vital observation.

PROPOSITION 3.

(a) *If $x \neq 1$ is in $\mathbb{N}$, then x is in* **M** *if and only if $x \equiv 0 \pmod 2$ and $x \equiv 1 \pmod 3$.*

(b) *If $x \neq 1$ is in* **M**, *where $x = 2^k w$ with w odd and $k \geq 3$, then x is not an atom of* **M**.

Proof. (a) ($\Rightarrow$) Suppose $x \neq 1$ is in $\mathbf{M}$. By definition, $x \equiv 4 \pmod 6$. Since then x is even, $x \equiv 0 \pmod 2$. Write $x - 4 = 6t$ for some t in $\mathbb{Z}$. Then $x - 1 = 6t + 3 = 3(2t + 1)$, so we have $x \equiv 1 \pmod 3$. ($\Leftarrow$) Suppose $x \neq 1$ is in $\mathbb{N}$ with $x \equiv 0 \pmod 2$ and $x \equiv 1 \pmod 3$. Now, by our second congruence, we have that $x - 1 = 3s$ for some s in $\mathbb{Z}$. Thus, $x - 4 = 3s - 3 = 3(s - 1)$. Since x is even, $x - 4$ is even, so 2 divides $s - 1$ and hence $x - 4 = 6q$ for some q in $\mathbb{Z}$. It follows that $x \equiv 4 \pmod 6$, completing the argument.

(b) Let $x \neq 1$ be in $\mathbf{M}$, where $x = 2^k w$ with w odd and $k \geq 3$. We show that $2^{k-2}w$ is also in $\mathbf{M}$. By (a), $x \equiv 1 \pmod 3$. It follows that

$$1 \equiv x \equiv 2^2(2^{k-2}w) \equiv 2^{k-2}w \pmod 3,$$

and since $k \geq 3$, $2^{k-2}w \equiv 0 \pmod 2$, which by (a) implies that $2^{k-2}w$ is in $\mathbf{M}$. So $x = (2^2)(2^{k-2}w)$ is a factorization of x in $\mathbf{M}$, and x is not an atom of $\mathbf{M}$. ■

We are now able to characterize the atoms of $\mathbf{M}$.

PROPOSITION 4. *An element x is irreducible in $\mathbf{M}$ if and only if*

(A) *$x = 2r$, where r is a positive odd number congruent to 2 modulo 3; and*

(B) *$x = 4s$, where $s = 1$ or s is a product of odd primes in $\mathbb{Z}$, all of which are congruent to 1 modulo 3.*

Moreover, none of the atoms of $\mathbf{M}$ are prime in $\mathbf{M}$.

Proof. ($\Rightarrow$) Let x be an atom of $\mathbf{M}$. Write $x = 2^k w$, where w is odd and $k \geq 1$. By Proposition 3(b), $k = 1$ or 2.

If $k = 1$, then $2w = x \equiv 1 \pmod 3$. Multiplying by 2 yields that $w \equiv 2 \pmod 3$. Thus, the condition in (A) is satisfied.

If $k = 2$ and $w = 1$, then clearly $x = 4$ is an atom. So suppose $w > 1$. Then $4w = x \equiv 1 \pmod 3$ and hence $w \equiv 1 \pmod 3$. Write $w = p_1 \cdots p_t$, where each p_i is an odd prime in $\mathbb{Z}$. Since $p_1 \cdots p_t \equiv 1 \pmod 3$, the number of primes p_i with $p_i \equiv 2 \pmod 3$ must be even. We proceed by contradiction. Suppose at least one of the p_i is congruent to 2 modulo 3. Then at least two of the primes must be congruent to 2 modulo 3; without loss of generality, we may assume $p_1 \equiv p_2 \equiv 2 \pmod 3$. Then

$$4w = 4p_1 \cdots p_t = (2p_1)(2p_2p_3 \cdots p_t).$$

Since $4w$ and $2p_1$ are both congruent to 1 modulo 3, $2p_2p_3 \cdots p_t$ must also be congruent to 1 modulo 3, and hence in $\mathbf{M}$ by Proposition 3(a). This contradicts the fact that x is an atom of $\mathbf{M}$. So no p_i is congruent to 2 modulo 3. Since no p_i can be divisible by 3 (else we would have $3 \mid w$), it follows that each p_i must be congruent to 1 modulo 3.

($\Leftarrow$) If $x = 2r$ is of Type A, then x is exactly divisible by 2 and hence must be an atom of $\mathbf{M}$. Let $x = 4s$ be an element of Type B. Using unique factorization in $\mathbb{Z}$, the only possible nontrivial factorization in $\mathbf{M}$ of x is of the form $x = 4s = (2u)(2v)$, where u and v are odd numbers that are products of odd primes all congruent to 1 modulo 3. Thus, both $2u$ and $2v$ are congruent to 2 modulo 3, which, by Proposition 3(a), contradicts the fact that they are elements of $\mathbf{M}$. Hence, $x = 4s$ is an atom of $\mathbf{M}$.

To verify the last statement, let $y = 2r$ be an atom of Type A and $z = 4s$ with $s \geq 1$ be an atom of Type B. We have

$$(2r)(4s) = (2rs)(4);$$

since $4 \in \mathbf{M}$, and since

$$\begin{aligned} rs \equiv 2 \pmod 3 \quad &\Rightarrow \quad rs = 3k + 2 \text{ for some } k \in \mathbb{Z} \\ &\Rightarrow \quad 2rs = 6k + 4 \text{ for some } k \in \mathbb{Z} \\ &\Rightarrow \quad 2rs \in \mathbf{M}, \end{aligned}$$

neither y nor z are prime, for both y and z divide $(2rs)(4)$, but neither divides $2rs$ or 4. This completes the proof. ■

On the elasticity of M

We have seen that Meyerson's monoid is not half-factorial. Given $x \in \mathbf{M}$, it is reasonable to ask how the lengths of various irreducible factorizations of x in $\mathbf{M}$ can differ. To explore this further, we will need a few more definitions. Let M be an atomic commutative cancellative monoid. Define, for $x \in M^*$,

$$L(x) = \text{the longest length of an irreducible factorization of } x \text{ in } M,$$

$$\ell(x) = \text{the shortest length of an irreducible factorization of } x \text{ in } M,$$

and

$$\rho(x) = \frac{L(x)}{\ell(x)} \in \mathbb{Q}_{\geq 1}.$$

The constant $\rho(x)$ is called the *elasticity* of x, and

$$\rho(M) = \sup\{\rho(x) \mid x \in M^*\}$$

is called the *elasticity* of M. If there exists an $x \in M^*$ such that $\rho(M) = \rho(x) = L(x)/\ell(x)$, then we say that the elasticity of M is *accepted*. A great deal of the recent mathematical literature devoted to nonunique factorizations focuses on elasticities and their calculations. Many such examples can be found in [**2**] and [**9**]. Prior to 2007, there were very few known examples in the literature of monoids with nonaccepted finite rational elasticity. None of these examples were elementary in nature.

EXAMPLE. By a previous Corollary, $\rho(\mathbf{H}) = 1$. In fact, the elasticity of any half-factorial monoid is 1.

EXAMPLE. Notice that it is possible for a monoid to have elasticity ∞. Let M be the arithmetical congruence monoid

$$M(6, 6) = 1, 6, 12, 18, 24, 30, \ldots = 6\mathbb{N} \cup \{1\}.$$

Clearly, 6 is an atom of $M(6, 6)$. For any integer $n \geq 2$, let $x_n = 6^n$. Notice that $x_n = (2^{n-1}3)(3^{n-1}2)$. Since each of $2^{n-1}3$ and $3^{n-1}2$ are exactly divisible by 6, each is an atom. Thus,

$$\rho(x_n) \geq \frac{n}{2},$$

and clearly $\rho(M(6, 6)) = \infty$.

Notice that a similar factorization technique works on any nonunit nonatom of $M(6, 6)$. Let $x = 2^k 3^m w$ be in $M(6, 6)$ with w relatively prime to both 2 and 3, and with $k, m \geq 2$. In $6\mathbb{N}$, we have

$$x = (2 \cdot 3^{m-1})(2^{k-1} \cdot 3 \cdot w),$$

and since each factor on the right-hand side is exactly divisible by 6, they are each atoms. Thus, any nonunit nonatom can be factored as a product of two irreducible elements of $M(6, 6)$. In general, an atomic monoid with this property is called *bifurcus* [**1**]. Hence, any bifurcus monoid has infinite elasticity.

We will now calculate the elasticity of the Meyerson monoid. We will need one further tool. A function f from $M \to \mathbb{Q}_{\geq 0}$ is called a *semi-length function* if

(a) $f(xy) = f(x) + f(y)$ for all x and y in M, and
(b) $f(x) = 0$ if and only if x is a unit of M.

The following fundamental result first appeared in [**2**].

PROPOSITION 5. *Let M be an atomic commutative cancellative monoid and f a semi-length function on M. If M has an irreducible element that is not prime, then*

$$\rho(M) \leq \frac{\sup\{f(y) \mid y \in \mathcal{A}(M) \text{ and } y \text{ not prime in } M\}}{\inf\{f(y) \mid y \in \mathcal{A}(M) \text{ and } y \text{ not prime in } M\}}.$$

Proof. Let $x \in M$ and suppose that $x = \alpha_1 \cdots \alpha_s = \beta_1 \cdots \beta_t$, where $s \geq t$ and each α_i and β_j is irreducible in M. Since the same number of prime factors (up to associates) appear in each factorization and $s/t \leq (s-k)/(t-k)$ for $1 \leq k \leq t-1$, we can assume without loss of generality that none of the irreducibles in the two factorizations of x are prime. For simplicity, set

$$m^* = \sup\{f(y) \mid y \in \mathcal{A}(M) \text{ and } y \text{ not prime in } M\}$$

and

$$m_* = \inf\{f(y) \mid y \in \mathcal{A}(M) \text{ and } y \text{ not prime in } M\}.$$

Using the semi-length function f, we have

$$f(x) = f(\alpha_1) + \cdots + f(\alpha_s) = f(\beta_1) + \cdots + f(\beta_t)$$

and hence

$$s \cdot m_* \leq f(\alpha_1) + \cdots + f(\alpha_s) = f(\beta_1) + \cdots + f(\beta_t) \leq t \cdot m^*.$$

This implies that $s/t \leq m^*/m_*$, completing the proof. ■

The proof of the following theorem first appeared in [**3**].

MEYERSON'S THEOREM. *The elasticity of* **M** *is*

$$\rho(\mathbf{M}) = 2.$$

Moreover, if $x \in$ **M**, *then $1 \leq \rho(x) < 2$ and hence the elasticity of* **M** *is not accepted.*

Proof. For each $x \in \mathbf{M}$ written as $2^k w$ where w is odd, set $f(x) = k$. Clearly, f is a semi-length function on **M**. By Proposition 4, $m^* = 2$ and $m_* = 1$, and so by Proposition 5, $\rho(\mathbf{M}) \leq m^*/m_* = 2$.

We now argue that $\rho(\mathbf{M}) = 2$. Consider the atoms $x = 2^2 \cdot 7$ and $y = 2 \cdot 7 \cdot 5$. For any positive integer c, set $t_c = (2 \cdot 7 \cdot 5)^{2c+1}$. By definition, t_c can be written as a product of $2c + 1$ atoms. Notice that it can also be written as

$$t_c = (2^2 \cdot 7)^c (2 \cdot 7^{c+1} \cdot 5^{2c+1}),$$

and since $2 \cdot 7^{c+1} \cdot 5^{2c+1} \equiv 4 \pmod 6$ and $7^{c+1} \cdot 5^{2c+1} \equiv 2 \pmod 3$, $2 \cdot 7^{c+1} \cdot 5^{2c+1}$ is an element of **M**. Moreover, since $2 \cdot 7^{c+1} \cdot 5^{2c+1}$ is exactly divisible by 2, it is an atom. Since $2^2 \cdot 7$ is an atom of Type B, t_c can be written as a product of $c+1$ atoms and

$$\rho(\mathbf{M}) \geq \frac{2c+1}{c+1}.$$

Since $\lim_{c\to\infty} \frac{2c+1}{c+1} = 2$, it follows that $\rho(\mathbf{M}) = 2$.

We now argue that no element $x \in \mathbf{M}$ has elasticity 2. Suppose $x \in \mathbf{M}$ has $\rho(x) = 2$. Suppose $x = \alpha_1 \cdots \alpha_{L(x)} = \beta_1 \cdots \beta_{l(x)}$ are irreducible factorizations of x in **M**. Since $L(x)/\ell(x) = 2$, we must have that each α_i is an irreducible of Type A and β_j is an irreducible of Type B. As a product of irreducibles of Type B, x has no odd factor that is congruent to 2 modulo 3; but, also, as a product of irreducibles of Type A, it must have an odd factor that is congruent to 2 modulo 3. Thus, we have a contradiction. ■

In conclusion, **M** is a simple example of an atomic monoid with elasticity 2 and the elasticity is **not accepted**. This verifies the contrast alluded to in the introduction, concerning the behavior of factorizations in **H** and **M**. We also note that readers interested in a further discussion of factorization properties can consult the excellent monograph [**13**].

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Summary Arithmetic sequences are among the most basic of structures in a discrete mathematics course. We consider here two particular arithmetic sequences:

$$1,\ 5,\ 9,\ 13,\ 17,\ \ldots \qquad (\mathbf{H})$$

and

$$4,\ 10,\ 16,\ 22,\ 28,\ \ldots. \qquad (\mathbf{M})$$

In addition to their additive definitions, these sequences are also multiplicatively closed. We show that both have multiplicative structures much different than that of the regular system of the integers. In particular, both fail the celebrated Fundamental Theorem of Arithmetic. While this is relatively easy to see, we will show that while factoring elements in the set **H** is fairly straightforward, factoring elements in **M** is much more complicated. This gives us a glimpse of how systems that fail the Fundamental Theorem of Arithmetic are studied and analyzed.

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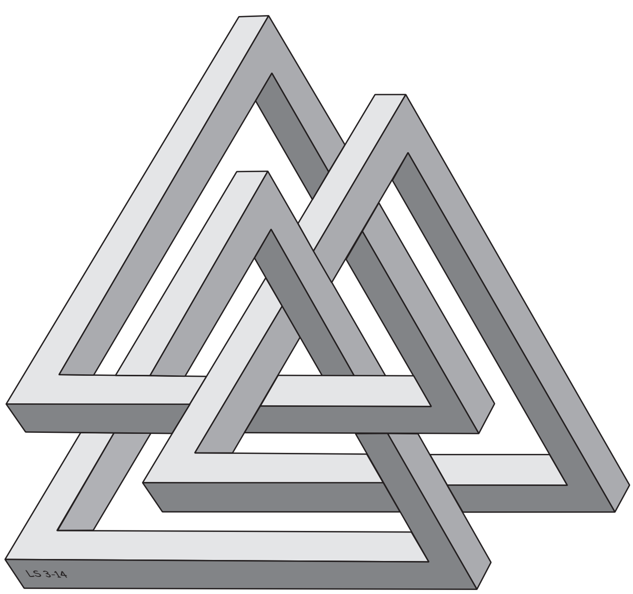

—Lee Sallows

Evangelista Torricelli and the "Common Bond of Truth" in Greek Mathematics

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In 1644, the Italian mathematician Evangelista Torricelli, best known as one of the two closest companions of Galileo at his death, published one of the most important works in the history of calculus that you've probably never heard of before. Torricelli's *Opera Geometrica* weighs in at almost 400 pages. One of the most important mathematical sections of the work, *de Dimensione Parabolae*, spans 84 pages and consists of just 22 propositions. Quite surprisingly for a mathematics book, each of these propositions states exactly the same conclusion:

A parabola is 4/3 *of the triangle having the same base and height.*

Setting aside for a moment just what Torricelli meant by the "base" and "height" of a parabola, for European mathematicians at the time the irony was that this result wasn't even original to Torricelli. It had been proved more than 1800 years previously in Archimedes' well-known *Quadrature of the Parabola*:

THOEREM (PROPOSITION 24 OF QP). *Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height [**3**, p. 251].*

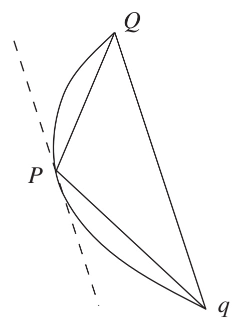

Figure 1 A parabola is 4/3 of the triangle having the same base and height.

Starting with a parabola passing through the three points Q, P, and q, the line segment Qq and the parabola enclose a region called a *segment* of the parabola, which we'll denote by par(qPQ). If P is now chosen to be the point on the parabola with tangent parallel to Qq, then P is called the *vertex* of par(qPQ). Likewise, Qq is the *base*, and the distance from P to Qq is the *height*. If $\triangle qPQ$ denotes the inscribed triangle, then Archimedes' result gives a relation between the areas of the two figures. In the proportional expression preferred by geometers at the time, it states:

$$\frac{\text{area}\,(\text{par}\,(qPQ))}{\text{area}\,(\triangle qPQ)} = \frac{4}{3} \tag{1}$$

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Equation (1) is one of the first nontrivial area or "quadrature" problems ever solved. It reduces the computation of the area of a segment of the parabola to the computation of the area of the inscribed triangle. Euclid I.42 and Euclid II.14 (or VI.13) then show how to find a square ("quadratus") with the same area as the given triangle.

But why would Torricelli's work, which simply revisits Archimedes' well-known result over and over again, be so important? The difference between each proposition is, of course, how he arrives there. Crucial to many of his proofs is the newfound (and controversial) analytic technique now called Cavalieri's Principle. With this technique, Torricelli can demonstrate in relatively short order the astounding result that Archimedes' quadrature of the parabola is, in fact, logically equivalent to nearly all of the major theorems of classical Greek geometry, including the other major results of Archimedes himself.

Why is this important? As any student of calculus will tell you, many of the best known applications of the integral first came to light in the works of Archimedes: In *Quadrature of the Parabola*, he solves the first area problem; in *On the Sphere and the Cylinder*, he finds a volume of revolution; in *On Spirals*, he finds the area of a region in polar coordinates; in *On the Equilibrium of Planes*, he finds centers of gravity. However, what isn't so clear in Archimedes' work is how these results relate to one another. In this paper we will show how Torricelli applied the full power of this new analysis to find a hidden, abstract connection between these works of Archimedes and, in the process, take a pivotal step on the road to the integral we know today.

Some facts about parabolas

We like to think that we know a thing or two about the curve $y = kx^2$ nowadays. But the classical Greek definition of the parabola gave these geometers some insights into parabolas that aren't always obvious with Cartesian coordinates. Archimedes lays out these results in the first few propositions of the *Quadrature of the Parabola*. We summarize them here.

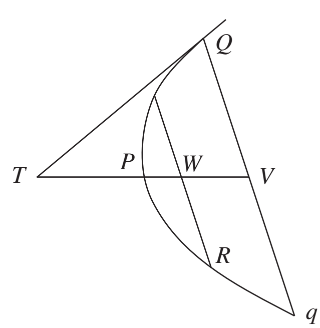

Figure 2 $PV/PW = Vq^2/WR^2$

LEMMA 1. (PROPOSITIONS 1–3 OF QP) *Let P, Q, R, and q be points on a parabola, and suppose that PV is parallel to its axis. Let Qq and WR, which intersect PV at V and W, respectively, be parallel to the tangent of the parabola at P. Suppose also that the tangent to the parabola at Q intersects PV at T. Then*

$$QV = Vq \tag{2}$$

$$PT = PV \tag{3}$$

$$\frac{PV}{PW} = \frac{Vq^2}{WR^2} \tag{4}$$

What do these results mean? As we know, the parabola is symmetric about its axis, so the axis will bisect any segment through the parabola and parallel to the tangent at the vertex. Equation (2) shows that any line PV parallel to the axis will likewise bisect linear segments Qq that are parallel to the tangent at P. (This explains why we call P the *vertex* of a segment with base parallel to the tangent at P.) Equation (3) gives the standard coordinate-free way to describe the tangent to a parabola, found in both Archimedes and Apollonius [**2**, p. 57]. Equation (4) is simply a proportional equivalent to our Cartesian definition of a parabola.

It is a challenge to prove these results using Cartesian coordinates. Archimedes himself gives no proofs, noting that they are "proved in the elements of conics" [**3**, p. 235], referring to a work that is now lost. Proofs are given in [**6**, p. 75–79].

In the *Quadrature of the Parabola*, Archimedes also proves two other results about parabolas and their tangents that will be very useful to Torricelli [**3**, pp. 237 and 246].

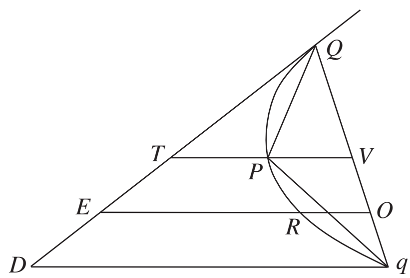

Figure 3 Lemmas 2, 3, and 5

LEMMA 2. (PROPOSITION 5 OF QP) *Suppose that the line Qq is the base of a segment of a parabola. Let EQ be the tangent at Q and let the line EO, passing through the parabola at R and Qq at O, be parallel to the axis of the parabola. Then*

$$\frac{QO}{Oq} = \frac{ER}{RO}.$$

LEMMA 3. (SEE THE PROOF OF PROPOSITION 17 OF QP) *Suppose now that PV is parallel to the axis of a parabola and that Qq is parallel to the tangent at P. Suppose that Dq is also parallel to the axis and that QD is tangent to the parabola at Q. Then*

$$\text{area}\,(\triangle qPQ) = \frac{1}{4}\,\text{area}\,(\triangle qDQ).$$

Lemma 3 is useful because, combined with equation (1), it shows that Archimedes' quadrature result is equivalent to a proportion involving the tangent triangle $\triangle qDQ$:

$$\frac{\text{area}\,(\text{par}\,(qPQ))}{\text{area}\,(\triangle(qDQ))} = \frac{1}{3}. \tag{5}$$

Torricelli will use without proof two additional lemmas about parabolas that aren't given in the *Quadrature of the Parabola*.

LEMMA 4. *Suppose that PV is parallel to the axis of a parabola and that Qq is parallel to the tangent of the parabola at P. Suppose R on the parabola, W on PV, and O on Qq are such that WR is parallel to the tangent at P and RO is parallel to PV. Then*

$$\frac{RO}{PV} = \frac{QO \cdot Oq}{Vq^2} = \frac{(QV + VO) \cdot Oq}{Vq^2}. \tag{6}$$

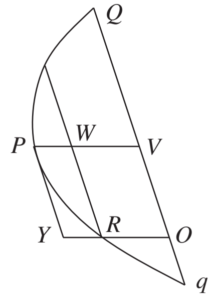

Figure 4 Lemma 4

Proof. Draw the tangent at P and let it intersect RO at Y. Then $PYRW$, $WROV$, and $PYOV$ are parallelograms, so by equation (4) and the equality of opposite sides of parallelograms,

$$\frac{VO^2}{Vq^2} = \frac{WR^2}{Vq^2} = \frac{PW}{PV} = \frac{YR}{YO} \iff 1 - \frac{YR}{YO} = 1 - \frac{VO^2}{Vq^2}$$
$$\iff \frac{RO}{YO} = \frac{Vq^2 - VO^2}{Vq^2}.$$

But $YO = PV$. Also,

$$Vq^2 - VO^2 = (Vq + VO)(Vq - VO) = (Vq + VO) \cdot Oq = (QV + VO) \cdot Oq$$

by equation (2), so equation (6) follows. ∎

LEMMA 5. *Suppose that Qq, PV, and RO are as in Lemma 4 and that the tangent to the parabola at Q intersects PV at T, RO at E, and the line Dq parallel to these lines through q at D. Then*

$$\frac{Dq}{ER} = \frac{DQ^2}{EQ^2}.$$

Proof. (See FIGURE 3.) The similarity of $\triangle QTV$ and $\triangle QDq$ and equations (2) and (3) show $Dq = 4\,PV$. So by Lemmas 2 and 4,

$$\frac{Dq}{ER} = \frac{4\,PV}{ER} = 4\,\frac{RO}{ER} \cdot \frac{PV}{RO} \qquad (\text{since } \frac{RO}{RO} = 1)$$
$$= 4\,\frac{Oq}{QO} \cdot \frac{Vq^2}{QO \cdot Oq}$$
$$= \frac{(2\,Vq)^2}{QO^2} = \frac{Qq^2}{QO^2}.$$

By the similarity of $\triangle qDQ$ and $\triangle OEQ$,

$$\frac{DQ}{EQ} = \frac{Qq}{QO},$$

and the result is proved. ∎

These insights on the parabola will be the key to understanding Torricelli's results, but we'll also need to be familiar with Archimedes' other works.

The sphere and the cylinder

In *On the Sphere and the Cylinder*, Archimedes solves the first nontrivial volume of revolution problem. He does this by inscribing the sphere inside of a cylinder with the same radius and height equal to the diameter of the sphere. As with the *Quadrature of the Parabola*, he establishes that the volumes of the two figures are related by a proportion [**3**, p. 43]:

$$\frac{\text{cylinder}}{\text{sphere}} = \frac{3}{2}.$$

This ratio between the sphere and the cylinder is the starting point for the proof in Proposition 13 of *de Dimensione Parabolae*. Torricelli begins by supposing that a parabola ABC is cut by a line AC. Let B denote the vertex of the corresponding segment of the parabola and suppose BH parallel to the axis of the parabola bisects AC. Now suppose that the segment AC is also the diameter of a semi-circle ANC with radius HN perpendicular to this diameter. Circumscribe a parallelogram AD around the segment par(ABC) of the parabola and a rectangle AE around the semi-circle. Finally, at an arbitrary point G on AC, draw a line FG parallel to BH, which intersects the parabola in I and AD in F, and another line GL parallel to the radius HN, which intersects the semi-circle in M and AE in L.

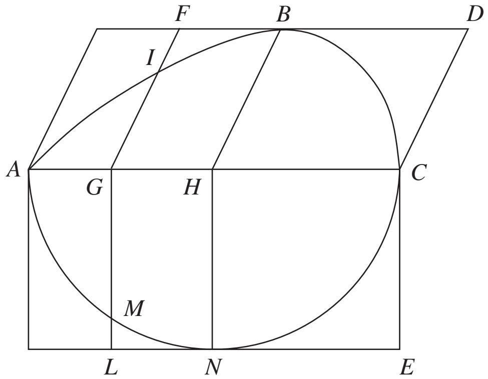

Figure 5 Semicircle *ANC* and segment *ABC* of a parabola

Next, suppose that the circle ANC and the rectangle AE are revolved around the axis AC. The result, of course, is a sphere inscribed in a cylinder. Since $FG = BH$ and $HN = GL = HA$, the following proportions are true at each point G on the axis AC:

$$\frac{FG}{GI} = \frac{BH}{GI} = \frac{HA^2}{CG \cdot GA} \qquad \text{(by Lemma 4)}$$

$$= \frac{HN^2}{GM^2} \qquad \text{(see the proof of Euclid II.14 or VI.13)}$$

$$= \frac{\pi\, GL^2}{\pi\, GM^2}.$$

The segments FG and GI are slices of the parallelogram and the parabola, respectively. Similarly, $\pi\, GL^2$ and $\pi\, GM^2$ are the respective areas of the corresponding slices of the cylinder and the sphere. Since the proportion is true at each point G on the axis AC, all Torricelli needs is a result that will allow him to move from a proportion which holds at any point G to a global claim about a proportion involving the figures in their entireties.

This was provided by Torricelli's mathematical contemporary Bonaventura Cavalieri, a Jesuati priest and lecturer in mathematics at the University of Bologna. "Cavalieri's Principle," as it eventually became known, would raise eyebrows today, and it was also somewhat controversial in his own time [**1**, p. 354]. But Cavalieri's works on indivisibles were "the most quoted sources (save for Archimedes) on geometric integration in the seventeenth century" [**4**, p. 123], and they were widely accepted throughout Europe by the 1650's [**1**, p. 355]. Because it was a welcome respite from the tedious method of exhaustion employed by Euclid and Archimedes for solving area and volume problems, it became a comfortable addition to the toolkits of many mathematicians raised on Euclidean proportions. Though Torricelli himself was initially sceptical of Cavalieri's work, in the latter half of *de Dimensione Parabolae* he thoroughly embraced it. In its most general form, Torricelli's interpretation of Cavalieri's Principle reads as follows.

LEMMA 6. (PROPOSITION 29 OF DE DIMENSIONE PARABOLAE) *If a first magnitude is to a second as a third is to a fourth, and thus however often it will have been pleasing, and if all the firsts and also all the thirds are proportional in the same order, then all the firsts together will be to all the seconds together as all the thirds together are to all the fourths together.*

In our case, all the "firsts" (that is, the segments FG) and all the "thirds" (that is, the circles with radius GL) are respectively the same, so the "in the same order" proportionality condition follows trivially. Hence, Torricelli concludes:

> *Therefore, all the firsts (namely, the parallelogram AD) will be to all the seconds (namely, to the parabola ABC) as all the thirds (that is, the cylinder) are to all the fourths together (clearly to the sphere).* [**12**, p. 61]

In other words, it follows from Cavalieri's Principle that

$$\frac{\text{area}\,(AD)}{\text{area}\,(\text{par}\,(ABC))} = \frac{\text{cylinder}}{\text{sphere}} = \frac{3}{2}.$$

Since $2\,\text{area}(\triangle ABC) = \text{area}(AD)$, equation (1) follows by taking reciprocals.

The importance of what has happened here shouldn't be overlooked: In just a few pages, Torricelli has used Cavalieri's Principle to demonstrate that the primary results in two major works of Archimedes are completely equivalent. However, he isn't finished.

On spirals

The Archimedean spiral is the locus of points obtained by revolving a ray around a central point O at a constant rate, while simultaneously moving away from that point at a constant rate along the ray. (In modern terms, the Archimedean spiral is the polar graph of the equation $f(\theta) = k\theta$.) In *On Spirals*, Archimedes shows that, given the circle whose radius is the same as the radius of the spiral after one full turn, the polar areas inside the two figures are again related by a proportion [**3**, p. 178]:

$$\frac{\text{area}\,(\text{spiral})}{\text{area}\,(\text{circle})} = \frac{1}{3}.$$

As before, Torricelli will show that this proportion is completely equivalent to Archimedes' result on the quadrature of the parabola.

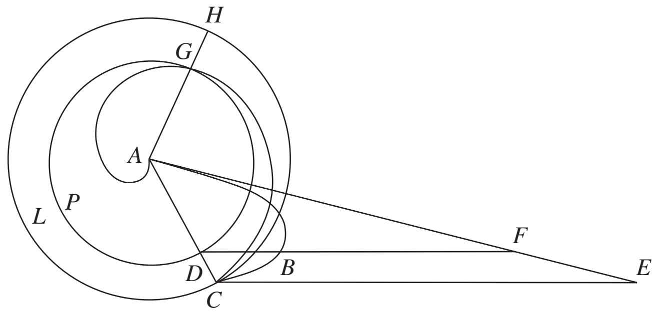

Figure 6 Spiral *AGC*, circle *CLHC*, and segment *ABC* of a parabola

To see how, consider the segment par(*ABC*) of a parabola determined by the line *AC*. Suppose that *AC* is also the radius of a circle *CLHC* with center at *A*, and consider the spiral *AGC* starting at *A* whose first turn ends at *C*. By Archimedes' result,

$$\frac{\text{area}\,(AGC)}{\text{area}\,(CLHC)} = \frac{1}{3}.$$

Next, consider the triangle $\triangle ECA$ formed by the tangent *AE* to the parabola at *A*, the line *CE* parallel to the axis of the parabola, and the line *AC*. As before, the key idea is to consider an arbitrary point *D* on the segment *AC*, establish a proportion (which in this case relates circular slices of the spiral region to corresponding slices of the parabola), and apply Cavalieri's Principle. To complicate matters, Torricelli actually works with the region inside the circle but *outside* the spiral (the area of which is in a ratio of 2 to 3 with the area of the circle itself, by *On Spirals*) and the "trilineum" *ABCE* which is inside the tangent triangle $\triangle AEC$ but outside the segment par(*ABC*).

Given *D* on *AC*, *AD* is the radius of a circle *DPGD* centered at *A* that intersects the spiral at, say, *G*. Thus, $AG = AD$. Now, consider the line *DF* parallel to *EC*, which intersects the parabola at *B* and *AE* at *F*. Since $\triangle ECA$ and $\triangle FDA$ are similar,

$$\frac{EC}{FD} = \frac{EA}{FA} = \frac{AC}{AD}.$$

Lemma 5, together with similarity and algebra, show that

$$\frac{EC}{FB} = \frac{EA^2}{FA^2} = \frac{EC^2}{FD^2} \iff \frac{EC}{FD} = \frac{FD}{FB}.$$

Since $AG = AD$, these strings of equalities imply that

$$\frac{FD}{FB} = \frac{AC}{AG}.$$

Given the point *G* on the spiral, now let *H* denote the point on the circle *CLHC* that lies on the ray *AG*. Since the ray *AC* is turning at a constant rate and the locus of the spiral is likewise moving along *AC* at a constant rate, the ratio of the length *AG* to the arclength arc(*CLH*) swept out on the circle *CLHC* in the same time interval is a constant independent of *G*. (In modern terms, this expresses the fact that $f(\theta)/\theta = k$ is constant on the Archimedean spiral.) Since the spiral arrives at the point *C* after one rotation, we have

$$\frac{AC}{\text{arc}(CLHC)} = \frac{AG}{\text{arc}(CLH)} \iff \frac{AC}{AG} = \frac{\text{arc}\,(CLHC)}{\text{arc}\,(CLH)}.$$

Note that the last ratio is independent of the radius, so

$$\frac{AC}{AG} = \frac{\text{arc}\,(CLHC)}{\text{arc}\,(CLH)} = \frac{\text{arc}\,(DPGD)}{\text{arc}\,(DPG)}.$$

Combining all of these results, we have

$$\frac{FD}{FB} = \frac{AC}{AG} = \frac{\text{arc}\,(CLHC)}{\text{arc}\,(CLH)} = \frac{\text{arc}\,(DPGD)}{\text{arc}\,(DPG)}.$$

As D ranges over AC, the lines FD fill out $\triangle AEC$ and the lines FB fill out the trilineum $ABCE$. Simultaneously, the arcs arc(DPG) fill out the *exterior* of the spiral region AGC and the arcs arc($DPGD$) fill out the entire circle $CLHC$. Thus, by Lemma 6 (Cavalieri's Principle) and *On Spirals*,

$$\frac{\text{area}\,(ABCE)}{\text{area}\,(\triangle AEC)} = \frac{\text{Exterior of spiral}}{\text{Circle}} = \frac{2}{3}.$$

Since the area outside the parabola is $2/3$ of $\triangle AEC$, the parabola must be $1/3$ of $\triangle AEC$. Equation (1) follows from Lemma 3 and, once again, in just a few pages, Torricelli has shown that two of Archimedes' majors works are completely equivalent.

On the equilibrium of planes

The familiar definition in calculus of the center of gravity (or center of equilibrium) of a plane region is, "the point P on which a thin plate of any given shape balances" [**11**, p. 600]. Thus, the center of a square is also its center of gravity because the square will balance at that point. But this is just one of several equivalent definitions used by Greek authors. As Sherman Stein points out, the center of gravity was also "the point common to all the lines on which the object balances" and "the point common to all the vertical lines through points of suspension" [**9**, p 16]. So if a square is suspended from the midpoint of one of its edges, it will be *at equilibrium* when the edge is horizontal, because the vertical line through this suspension point and the center of gravity is perpendicular to the edge itself.

Centers of gravity (or equilibrium) are the primary concern in the two books of Archimedes' *On the Equilibrium of Planes*. In Book I, he shows that the center of gravity of a triangle lies on the intersection of any two median lines. In Book II, he uses this result and the method of exhaustion to show that the center of gravity of a segment of a parabola lies on the axis at a point $3/5$ of the length of the axis from the vertex. Archimedes' "Law of the Lever" (Propositions 6 and 7 of Book I of *Plane Equilibrium*), which is his key result for finding centers of gravity, will also be crucial for Torricelli. It shows that two magnitudes A and B centered at points E and D on a beam will balance when the fulcrum is placed at point C such that DC and CE are reciprocally proportional—that is, $A/B = DC/CE$. So, for instance, if two squares of mass A and B are suspended from a balance beam ED by a vertical line through the midpoint of an edge, then the entire system will be in equilibrium when the fulcrum of the beam is at a point C (the center of equilibrium) such that $A/B = DC/CE$ and the edges of suspension of the squares are horizontal. (See FIGURE 7.)

Torricelli needs two additional results about centers of gravity that aren't found in Archimedes. First, he needs to know how to suspend a right triangle at equilibrium from its hypotenuse. It follows from Archimedes' result that the center of gravity of a triangle lies at a point that divides each of the median lines in a ratio of 2 to 1. (See [**8**, p. 28].) Now, let $\triangle CAB$ be right with legs AB and AC. Suppose that G is its

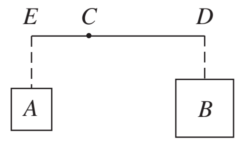

Figure 7 Squares A and B are at equilibrium when $A/B = DC/CE$.

center of gravity and that H is the midpoint of CA. Now suppose that the triangle is suspended from a point F on its hypotenuse, so that AB is horizontal. Then CA and GF are vertical, so $\triangle BHC$ and $\triangle BGF$ are similar. Since $BG = 2GH$, similarity implies the triangle will be suspended at equilibrium from a point F such that $CF = 2BF$.

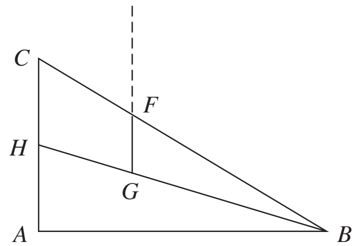

Figure 8 Right triangle $\triangle CAB$ is at equilibrium when $CF = 2\ BF$.

Second, Torricelli needs to know that the center of gravity of a cone lies on its axis at a point $1/4$ of the distance from the base to the vertex. Based on comments in *the Method* [**3**, App. p. 15] and elsewhere, scholars agree that Archimedes produced a work on the centers of gravity of solids, and this was one of his results [**7**]. Of course, *the Method* wasn't known in the seventeenth century, but Frederico Commandino, one of the earliest translators of Archimedes' works into Latin, had already re-established this result in his 1565 work *Liber de Centro Gravitatis Solidorum* [**5**, p. 28].

With these results in hand, consider a segment par(ABC) of a parabola with axis BE, where Torricelli now also assumes that AC and BE are perpendicular with AC horizontal and BE vertical. His proof (Proposition 14 of [**12**]) again begins with the trilineum $FABC$ contained within the tangent triangle $\triangle FAC$ of par(ABC), where FA is parallel to BE and hence vertical. (See FIGURE 9.) His appeal to Cavalieri's Principle is different—he now concludes something about the centers of gravity of corresponding objects—but the method of proof should be familiar. If LN is a slice of $\triangle FAC$ parallel to AF that intersects the parabola at M, then Lemma 5 together with similarity and algebra show that

$$\frac{FA}{LM} = \frac{FC^2}{LC^2} = \frac{FA^2}{LN^2} = \frac{\pi FA^2}{\pi LN^2} \iff \frac{\pi LN^2}{LM} = \pi FA.$$

Since πFA is independent of N, at each point N along AC there is a constant proportion between LM and πLN^2. But πLN^2 is the area of a slice of the right cone having base with radius FA, axis AC, and vertex C, and LM is the corresponding slice on the trilineum $FABC$. Torricelli concludes that

> *Therefore, since proportional magnitudes of two types are suspended from the same points ... all the magnitudes of the first type together (that is, all the lines of the trilineum [FABC], or the trilineum itself) will have the same point of equilibrium as all the magnitudes of the second type together (that is, all the circles of the cone ... or the cone itself)* [**12**].

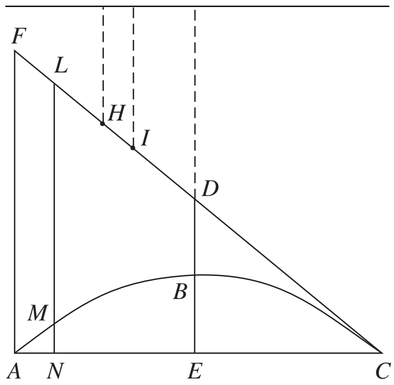

Figure 9 Trilineum $FABC$ is at equilibrium when $FH = \frac{1}{4}FC$.

So by Commandino's result, the trilineum $FABC$ will be suspended at equilibrium from a vertical line through the point H such that $FH = \frac{1}{4}FC$. But two other related points of suspension can also be found from the results given above: First, the entire triangle $\triangle FAC$ will be suspended at equilibrium from a vertical line through a point I such that $FI = \frac{1}{3}FC$. Second, since the center of gravity of a parabola lies on its axis and the axis DE of the parabola is perpendicular to the segment AC, equation (2) and the similarity of $\triangle FAC$ and $\triangle DEC$ show that par(ABC) will be suspended at equilibrium from a vertical line through the point D such that $FD = \frac{1}{2}FC$. Since $IH = FI - FH = \frac{1}{12}FC$ and $ID = FD - IF = \frac{1}{6}FC$, it follows that $ID/HI = 2$.

But $\triangle FAC$ consists of the trilineum $FABC$ and the segment par(ABC) taken together, so by the Law of the Lever, the center of gravity I of $\triangle FAC$ will also be the point where HI and ID are reciprocally proportional to area($FABC$) and area(par(ABC)). That is,

$$\frac{\text{area}\,(FABC)}{\text{area}\,(\text{par}\,(ABC))} = \frac{ID}{HI} = 2.$$

Consequently,

$$\frac{\text{area}\,(\triangle FAC)}{\text{area}\,(\text{par}\,(ABC))} = \frac{\text{area}\,(FABC) + \text{area}\,(\text{par}\,(ABC))}{\text{area}\,(\text{par}\,(ABC))} = 2 + 1 = 3.$$

Equation (1) again follows from Lemma 3.

Further results

Aficionados of Archimedes's results will be expecting a proof that makes use of his other major work related to calculus, *On Conoids and Spheroids*. Here, Archimedes finds formulas for the volumes of revolution obtained from the conic sections. Torricelli doesn't provide such a result, but he does prove equation (1) using another result about solids of revolution: Euclid XII.10, which shows that any cone is 1/3 of its circumscribing cylinder [**1**, p. 356]. He also proves equation (1) using the Pappus–Guldin Theorem [**12**, Proposition 19]. Altogether, in the second half of *de Dimensione Parabolae*, Torricelli provides 11 different proofs of Archimedes' area formula that make use of Cavalieri's method of indivisibles. This includes a restatement of Archimedes' original proof, which uses indivisibles and infinite series instead of Archimedes' painstaking method of exhaustion [**4**, p. 184]. As Torricelli himself notes,

> *In fact, with the principal theorems of the ancients assumed (they may concern very diverse matters themselves), as much of Euclid as of Archimedes, it is amazing that from every single one of these the quadrature of the parabola is able to be elicited with so little trouble and vice versa, as if there were a certain common bond of truth* [**12**].

Today we call that "common bond of truth" the integral, and we take it for granted that we can summarize and extend hundreds of pages of Greek deductions on volumes of revolution, polar areas, and centers of gravity in a handful of integral formulas. But Torricelli didn't have the benefit of this insight. What's truly amazing about his work is that he was able to see all of these connections using only the language of Euclidean geometry and Cavalieri's method of indivisibles. In doing so, he even surpassed Cavalieri himself. Indeed, because Cavalieri's work was so obscure and Torricelli's work was so compelling, it was Torricelli who most mathematicians turned to in order to understand the method of indivisibles [**1**, p. 353]. For those of us brought up to solve calculus problems algebraically, Torricelli's geometrical approach is almost unintelligible. But mathematical contemporaries such as John Wallis (who was read by both Newton and Leibniz) were brought up in a geometrical tradition, and they used Torricelli's work as a gateway to the study of infinitesimally small quantities. Though Wallis and others would ultimately see that "the necessary summations could be carried out arithmetically rather than geometrically" [**10**, p. xvi], Torricelli's *de Dimensione Parabolae* is still a vital step in the evolution of the analysis we know today.

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Summary In 1664, Evangelista Torricelli published his *Opera Geometrica*, one of the most important—yet most unheralded—publications in the history of integral calculus. In the chapter *de Dimensione Parabolae*, Torricelli uses the newfound analytic techniques of Bonaventura Cavalieri to prove, among other things, that all of the major geometrical works of Archimedes—*Quadrature of the Parabola*, *On the Sphere and the Cylinder*, *On Spirals*, and *On the Equilibrium of Planes*—are joined by a "common bond of truth." In this article, we show how Torricelli establishes this connection and discuss briefly the impact it had on subsequent mathematicians such as John Wallis.

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Outer Median Triangles

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The medians are special

A *median* of a triangle is a line segment that connects a vertex of the triangle to the midpoint of the opposite side. The three medians of a triangle interact nicely with each other to yield the following properties:

(a) The medians intersect in a point interior to the triangle, called the *centroid*, which divides each of the medians in the ratio 2 : 1.

(b) The medians form a new triangle, called the *median triangle*.

(c) The area of the median triangle is 3/4 of the area of the given triangle in which the medians were constructed.

(d) The median triangle of the median triangle is similar to the given triangle with the ratio of similarity 3/4.

When we say, as in (b), that "three line segments *form a triangle*" we mean that there exists a triangle whose sides have the same lengths as the line segments.

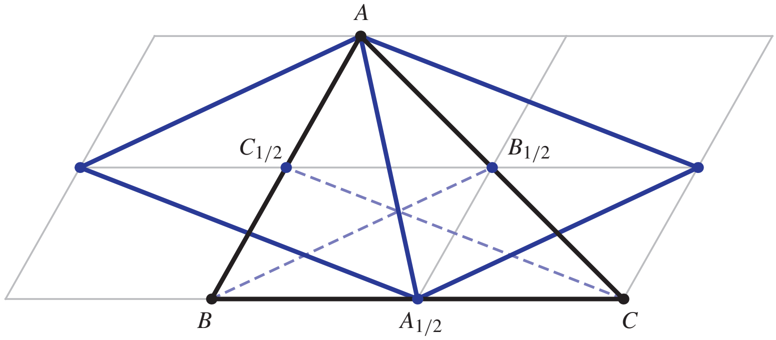

Figure 1 A "proof" of Properties (b) and (c)

Proving Property (a) is a common exercise. We provide "proofs without words" of Properties (b), (c), and (d) in FIGURES 1 and 2. Different proofs can be found in [**9**] and at [**17**]. Note that Property (b) fails for other equally important triples of cevians of a triangle; for example, as shown in [**2**], we cannot always speak about a triangle formed by bisectors or altitudes.

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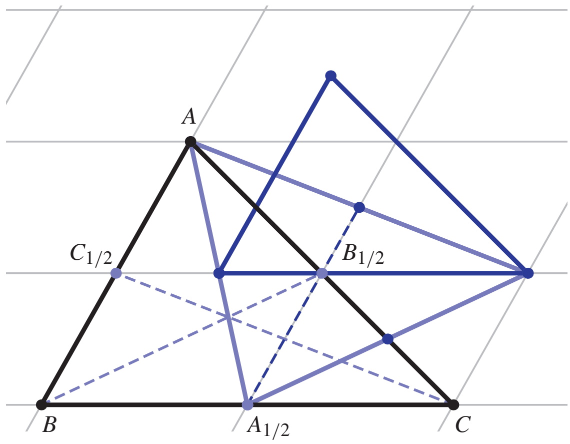

Figure 2 A "proof" of Property (d)

History Before introducing outer analogues of the medians and the median triangle, we reflect on the history of the above properties. Property (a) was proved by Archimedes of Syracuse, as Proposition 14 in Equilibrium of Planes, Book I; see [**8**], [**1**, Subsection 10.7.2], or [**16**, p. 86]. By the 19th century, Property (a) was a common proposition accompanying Euclid's Elements [**4, 14**]. Interestingly, in [**14**] and in several other books of this period, the term "median triangle" of ABC means, in our notation, the triangle $A_{1/2}B_{1/2}C_{1/2}$. Nowadays, this triangle is called the *medial triangle* of ABC.

The first usage of the current meaning of the median triangle that we found is in [**11**, Ch. XVI, §473]. However, in his 1887 paper [**13**], Mackay proves Property (b) in his §6 without explicitly stating it. He attributes his §6 to [**15**], but we could not find it there.

Furthermore, Property (c) appears as [**13**, §8(c)] and Property (d) as [**13**, §8(a)]. Mackay points out that [**13**, §8(a)] is proved in [**10**] as a solution to a problem proposed in [**12**]. Finally, Mackay believed that his [**13**, §8(c)] was new.

The medians are not alone

A median of a triangle is just a special *cevian*; a cevian is a line segment joining a vertex of a triangle to a point on the opposite side. Are there other triples of cevians from distinct vertices of a triangle that share the essential features of Properties (a), (b), (c), and (d)?

Some natural candidates for such cevians are suggested by the "median grid" already encountered in FIGURES 1 and 2. In FIGURES 3 and 4, we show more of this grid with the cevians that in some sense most resemble the medians. The labeling of the points on the line BC in FIGURE 4 originates from BC being considered as a number line with 0 at B and 1 at C. More precisely, for $\rho \in \mathbb{R}$, the point A_ρ is the point on the line BC that satisfies $\overrightarrow{BA_\rho} = \rho\,\overrightarrow{BC}$. The points on the lines AB and BC are labeled similarly.

As indicated in FIGURE 3, the cevians in the triple

$$\big(BB_{-1/2}, AA_{1/2}, CC_{3/2}\big) \tag{1}$$

are concurrent, with the understanding that three line segments are concurrent if the lines determined by them are concurrent. In addition to the triple of cevians in (1)

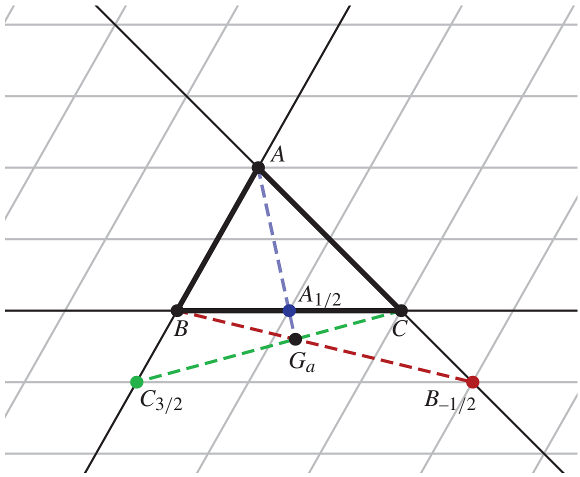

Figure 3 The "median grid"; the median from A, and one outer median from each B and C

we will consider two more triples of concurrent cevians that are symmetrically placed with respect to the other sides:

$$\left(CC_{-1/2}, BB_{1/2}, AA_{3/2}\right), \quad \left(AA_{-1/2}, CC_{1/2}, BB_{3/2}\right). \tag{2}$$

All three triples are shown in FIGURE 4.

That the triples in (1) and (2) are really concurrent follows from Ceva's theorem, which in our notation reads as:

CEVA'S THEOREM [**5**, p. 220]. *With $\rho, \sigma, \tau \in \mathbb{R}$, the cevians AA_ρ, CC_σ, BB_τ are concurrent if and only if*

$$\rho\sigma\tau - (1-\rho)(1-\sigma)(1-\tau) = 0. \tag{3}$$

Equation (3) defines a surface in $\rho\sigma\tau$-space; see FIGURE 10, below. We call it the *Ceva surface*. It will appear prominently in what follows.

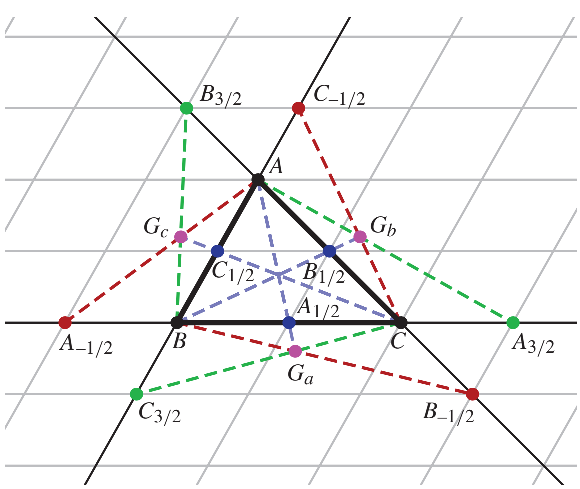

Figure 4 The "median grid"; three medians and six outer medians

Since the cevians $AA_{-1/2}$, $AA_{3/2}$, $BB_{-1/2}$, $BB_{3/2}$, $CC_{-1/2}$, $CC_{3/2}$ play the leading roles in this note and because of their proximity to the medians on the "median grid," we call them *outer medians*. Thus, for example, associated to vertex A we have one median, $AA_{1/2}$, and two outer medians, $AA_{-1/2}$ and $AA_{3/2}$. See FIGURE 4.

We find it quite remarkable that all four properties of the medians listed in the opening of this note hold for the three triples displayed in (1) and (2), each of which consists of a median and two outer medians originating from distinct vertices.

(A) The median and two outer medians in each of the triples in (1) and (2) are concurrent.

(B) The median and two outer medians in the triples in (1) and (2) form three triangles. We refer to these three triangles as *outer median triangles* of ABC; see FIGURE 5.

(C) The area of each outer median triangle of ABC is 5/4 of the area of ABC.

(D) For each outer median triangle, one of its outer median triangles is similar to the original triangle ABC with the ratio of similarity 5/4.

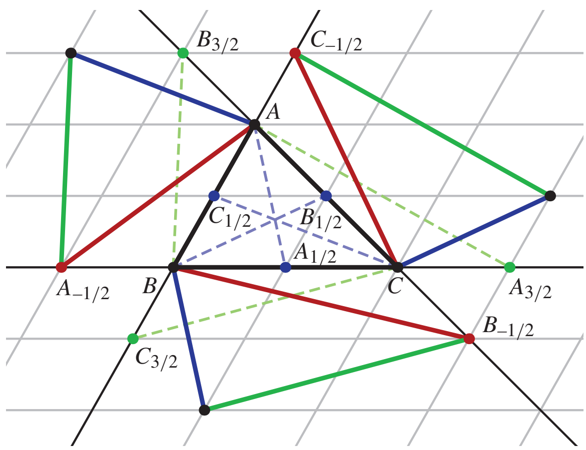

Figure 5 Three outer median triangles of *ABC*

As we have already mentioned, Property (A) follows from Ceva's theorem. FIGURES 6 and 7 offer "proofs without words" of Properties (B), (C), and (D).

We point out that the concurrency points G_a, G_b, G_c (see FIGURE 4) divide the corresponding outer medians in the ratio 2 : 3, that is, for example,

$$BG_a : G_aB_{-1/2} = CG_a : G_aC_{3/2} = 2 : 3.$$

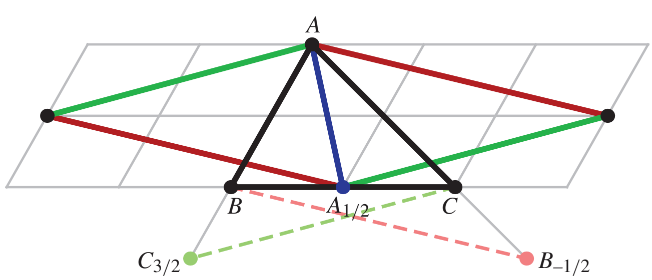

Figure 6 A "proof" of Properties (B) and (C)

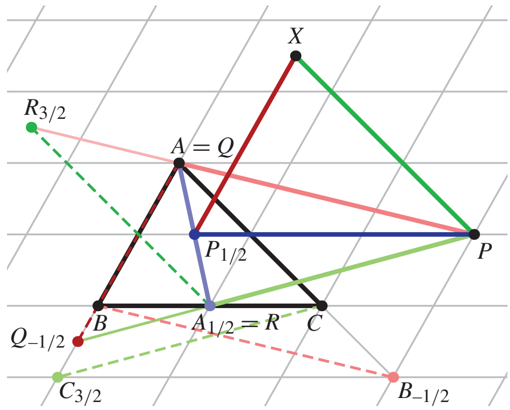

Figure 7 A "proof" of Property (D)

Similarly, it can be shown that the concurrency points divide the corresponding medians in the ratio 6 : 1, for example, $AG_a : A_{1/2}G_a = 6 : 1$. Computing these ratios is an exercise in vector algebra.

Are the medians and the outer medians alone?

We motivated our study of outer medians by their special position on the "median grid." However, the above four properties could very well hold for other triples of cevians. Is it then the case that the median triangle and the three outer median triangles are truly special?

The concurrency of three cevians is characterized by equation (3), which was used to justify the claim in Property (A). Next, akin to Property (B) and with no requirement, for the moment, that the cevians be concurrent, we look for a sufficient condition under which three cevians form a triangle.

Property (B) We first answer the following question: For which $\rho, \sigma, \tau \in \mathbb{R}$ does there exist a triangle with sides that are congruent and parallel to the cevians AA_ρ, BB_σ and CC_τ, independent of the triangle ABC in which they are constructed?

With $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$ and $\mathbf{c} = \overrightarrow{AB}$, we have

$$\overrightarrow{AA_\rho} = \mathbf{c} + \rho\,\mathbf{a}, \quad \overrightarrow{BB_\sigma} = \mathbf{a} + \sigma\,\mathbf{b} \quad \text{and} \quad \overrightarrow{CC_\tau} = \mathbf{b} + \tau\,\mathbf{c}.$$

Then, a necessary and sufficient condition for the existence of a triangle with sides that are congruent and parallel to the line segments AA_ρ, BB_σ, and CC_τ is that one of the following four vector equations is satisfied:

$$\overrightarrow{AA_\rho} \mathbin{\hat{\pm}} \overrightarrow{BB_\sigma} \pm \overrightarrow{CC_\tau} = \mathbf{0}. \tag{4}$$

We put a special sign ˆ above the first ± to be able to trace this sign in the calculations that follow. Substituting $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ in (4), we get

$$\big(-1 \mathbin{\hat{\pm}} 1 + \rho \mp \tau\big)\mathbf{a} + \big(-1 \mathbin{\hat{\pm}} \sigma \pm 1 \mp \tau\big)\mathbf{b} = \mathbf{0}. \tag{5}$$

Using the linear independence of **a** and **b** and choosing both + signs in (4), it follows from (5) that $\rho = \sigma = \tau$. Choosing the first sign in (4) to be + and the second to be −, we get that $\rho = -\tau$, $\sigma = 2 - \tau$. Choosing the first sign in (4) to be − and the second to be $+$, we get $\rho = 2 - \sigma$, $\tau = -\sigma$; and choosing both $-$ signs in (4), we get $\sigma = -\rho$, $\tau = 2 - \rho$. Thus, we have identified four sets of parameters (ρ,σ,τ) for which, independent of ABC, there exists a triangle, possibly degenerate, with sides that are congruent and parallel to the cevians AA_ρ, BB_σ, and CC_τ:

$$(\xi,\xi,\xi), \quad (2-\xi,\xi,-\xi), \quad (-\xi,2-\xi,\xi), \quad (\xi,-\xi,2-\xi), \quad \xi \in \mathbb{R}. \tag{6}$$

The only concern here is that the cevians AA_ρ, BB_σ, and CC_τ might be parallel. However, the condition for the cevians to be parallel is easily established as follows. Since the vector $\overrightarrow{CC_\tau}$ is nonzero, we look for λ, $\mu \in \mathbb{R}$ such that

$$\mathbf{c} + \rho\,\mathbf{a} = \lambda(\mathbf{b} + \tau\,\mathbf{c}) \quad \text{and} \quad \mathbf{a} + \sigma\,\mathbf{b} = \mu(\mathbf{b} + \tau\,\mathbf{c}). \tag{7}$$

Substituting $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ in (7) and using the linear independence of $\mathbf{a}$ and $\mathbf{b}$, we get from the first equation $\lambda = 1/(\tau - 1)$, $\rho = 1/(1-\tau)$ and from the second equation $\mu = -1/\tau$, $\sigma = 1 - 1/\tau$. Hence, the line segments AA_ρ, BB_σ, and CC_τ are parallel if and only if

$$\rho = \frac{1}{1-\xi}, \quad \sigma = 1 - \frac{1}{\xi}, \quad \tau = \xi, \quad \xi \in \mathbb{R} \setminus \{0, 1\}. \tag{8}$$

Thus, to avoid degeneracy of triangles with cevian sides corresponding to triples in (6) such as, for example, $(-\xi, 2-\xi, \xi)$, we must exclude the values of the parameter ξ that solve $-\xi = 1/(1-\xi)$. This, in turn, shows that the triples (ρ,σ,τ) for which there exists a non-degenerate triangle with sides that are congruent and parallel to the cevians AA_ρ, BB_σ, and CC_τ must belong to one of the following four sets:

$$\begin{aligned}
\mathbb{D} &= \big\{(\xi,\xi,\xi) : \xi \in \mathbb{R}\big\},\\
\mathbb{E} &= \big\{(2-\xi,\xi,-\xi) : \xi \in \mathbb{R}\backslash\{-\phi^{-1},\phi\}\big\},\\
\mathbb{F} &= \big\{(-\xi,2-\xi,\xi) : \xi \in \mathbb{R}\backslash\{-\phi^{-1},\phi\}\big\},\\
\mathbb{G} &= \big\{(\xi,-\xi,2-\xi) : \xi \in \mathbb{R}\backslash\{-\phi^{-1},\phi\}\big\},
\end{aligned}$$

where $\phi = (1+\sqrt{5})/2$ denotes the golden ratio.

The diagonal of the $\rho\sigma\tau$-space provides a geometric representation of the set $\mathbb{D}$. The other three sets are represented by straight lines with two points removed. All four lines are shown in FIGURE 10, together with the Ceva surface.

Generalized median and outer median triangles As we have just seen, the cevians associated with the triples in the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ are guaranteed to form triangles; that is, they satisfy a property analogous to Property (B). The most prominent representatives of triangles originating from the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ are the median and outer median triangles, which all correspond to the value $\xi = 1/2$. Therefore, for a fixed ξ, the triangle associated with the triple (ξ,ξ,ξ) in $\mathbb{D}$ we call *ξ-median triangle*, and the triangles associated with the corresponding triples in $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ we call *ξ-outer median triangles*. In FIGURES 8 and 9, we illustrate these triangles with $\xi = 1/\phi$, the reciprocal of the golden ratio.

Next, we explore whether the ξ-median and ξ-outer median triangles have properties analogous to (C) and (D).

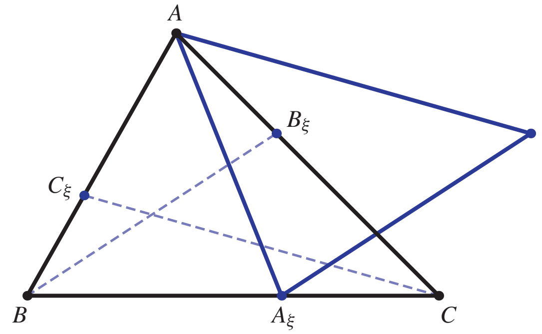

Figure 8 The ξ-median triangle with $\xi = \phi^{-1}$

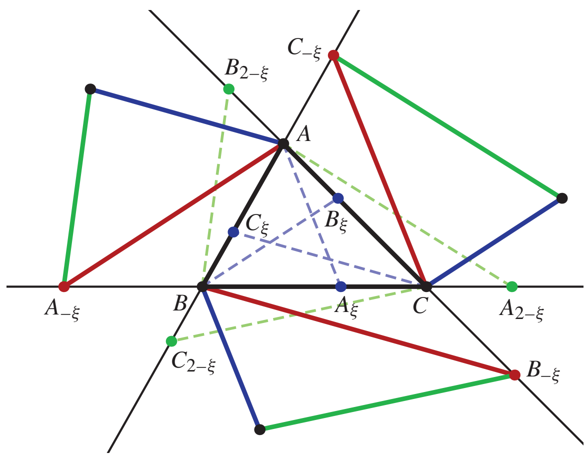

Figure 9 Three ξ-outer median triangles with $\xi = \phi^{-1}$

Property (C) First, we recall two classical formulas, which seem to be custom made for our task.

Heron's formula [**5**, 1.53], gives the square of the area of a triangle, Δ^2, in terms of its sides a, b, c:

$$\Delta^2 = s(s-a)(s-b)(s-c), \qquad \text{where} \qquad s = \frac{1}{2}(a+b+c).$$

Substituting s and simplifying yields

$$\Delta^2 = \frac{1}{16}\Big(2\big(a^2b^2 + b^2c^2 + c^2a^2\big) - \big(a^4 + b^4 + c^4\big)\Big).$$

Stewart's theorem [**6**, Section 1.2, Exercise 4], gives the square of the length of a cevian in terms of the squares of the sides of ABC:

$$(AA_\rho)^2 = \rho(\rho - 1)a^2 + \rho b^2 + (1-\rho)c^2.$$

Similar formulas hold for $\left(BB_\sigma\right)^2$ and $\left(CC_\tau\right)^2$. In matrix form, these three equations are:

$$\begin{bmatrix} (CC_\tau)^2 \\ (BB_\sigma)^2 \\ (AA_\rho)^2 \end{bmatrix} = \begin{bmatrix} \tau & 1-\tau & \tau(\tau-1) \\ 1-\sigma & \sigma(\sigma-1) & \sigma \\ \rho(\rho-1) & \rho & 1-\rho \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}. \tag{9}$$

We denote the 3×3 matrix in (9) by $M(\rho, \sigma, \tau)$. The idea of using this matrix is due to Griffiths [**7**]. It was further explored in [**3**, Section 3].

Now it is clear how to proceed to verify the property analogous to (C): use triples from the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ to get expressions for the squares of the corresponding cevians, substitute these expressions in Heron's formula, and simplify. However, this involves simplifying an expression with 36 additive terms, quite a laborious task for a human but a perfect challenge for a computer algebra system like *Mathematica*. We first define Heron's formula as a *Mathematica* function (we call it `HeronS`) operating on the *triples* of squares of the sides of a triangle and producing the square of the area:

```
In[1]:= HeronS[ {x_, y_, z_}] := 1/16 (2 (x*y + y*z + z*x) - (x^2+y^2+z^2))
```

Next, we define in *Mathematica* the matrix function `M` as in (9):

```
In[2]:= M[{ρ_, σ_, τ_}] := { { τ, 1−τ, −(1−τ)*τ },
                             { 1−σ, −(1−σ)*σ, σ },
                             { −(1−ρ)*ρ, ρ, 1−ρ } }
```

To verify the property analogous to (C) for ξ-median triangles, we put the newly defined functions in action by calculating the ratio between the squares of the area of the ξ-median triangle and the original triangle. *Mathematica*'s answer is instantaneous:

```
In[3]:= Simplify[HeronS[M[{ξ,ξ,ξ}].{x,y,z}]/HeronS[{x,y,z}]]

Out[3]= (1-ξ+ξ^2)^2
```

This "proves" that the ratio of the areas depends only on ξ, and that the ratio is exactly $1 - \xi + \xi^2$. Further, for one of the ξ-outer median triangles, we have

```
In[4]:= Simplify[HeronS[M[{2-ξ,ξ,−ξ}].{x,y,z}]/HeronS[{x,y,z}]]

Out[4]= (1+ξ-ξ^2)^2
```

"proving" that the area of the triangle formed by the cevians $AA_{2-\xi}$, BB_ξ, $CC_{-\xi}$ is $|1 + \xi - \xi^2|$ of the area of the original triangle ABC. The other two ξ-outer median triangles yield the same ratio. In summary, *Mathematica* has confirmed that the ξ-median and the three ξ-outer median triangles all have the property analogous to (C).

Property (D) The verification of the property analogous to (D) is simpler. For a ξ-median triangle, following [**7**], we just need to calculate the square of the matrix $M(\xi, \xi, \xi)$, which turns out to be $\left(1 - \xi + \xi^2\right)^2 I$. This confirms that the ξ-median triangle of the ξ-median triangle is similar to the original triangle with the ratio of similarity $1 - \xi + \xi^2$.

Similarly, for a ξ-outer median triangle corresponding to a triple in $\mathbb{E}$, we calculate the square of the matrix $M(2 - \xi, \xi, -\xi)$, which turns out to be $\left(1 + \xi - \xi^2\right)^2 I$; this confirms that one of the ξ-outer median triangles of this ξ-outer median triangle is similar to the original triangle with the ratio of similarity $|1+\xi-\xi^2|$. In contrast, for a ξ-outer median triangle corresponding to a triple in $\mathbb{F}$, to get a triangle similar to the original triangle we need to calculate its ξ-outer median triangle corresponding to a triple in $\mathbb{G}$. This amounts to multiplying the matrices

$$M(\xi,-\xi,2-\xi)M(-\xi,2-\xi,\xi)=\big(1+\xi-\xi^2\big)^2 I.$$

Likewise, for a ξ-outer median triangle corresponding to a triple in $\mathbb{G}$, we calculate its ξ-outer median triangle corresponding to a triple in $\mathbb{F}$ and obtain the same result.

Concurrency comes to the rescue All these calculations indicate that, after all, the median and outer median triangles are facing stiff competition from their ξ-triangles generalizations. However, property (A) comes to the rescue of the median and outer median triangles at this point. We want the triples of cevians corresponding to the triples in $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ to be concurrent as well. So which of these triples satisfy Ceva's condition (3)? Or, geometrically, what is the intersection of the lines and the Ceva surface in FIGURE 10? First, we substitute $\rho=\sigma=\tau=\xi$ in (3), which yields $\xi^3-(1-\xi)^3=0$, whose only real solution is $\xi=1/2$. The corresponding cevians are the medians. To intersect $\mathbb{E}$ with the Ceva surface, we substitute $(2-\xi,\xi,-\xi)$ in (3), obtaining $-\xi^2(2-\xi)-(1+\xi)(\xi-1)(1-\xi)=0$, which is equivalent to $(\xi-\phi)(\xi+\phi^{-1})(2\xi-1)=0$. Since $\xi\notin\{\phi,-\phi^{-1}\}$, the only solution is $\xi=1/2$,

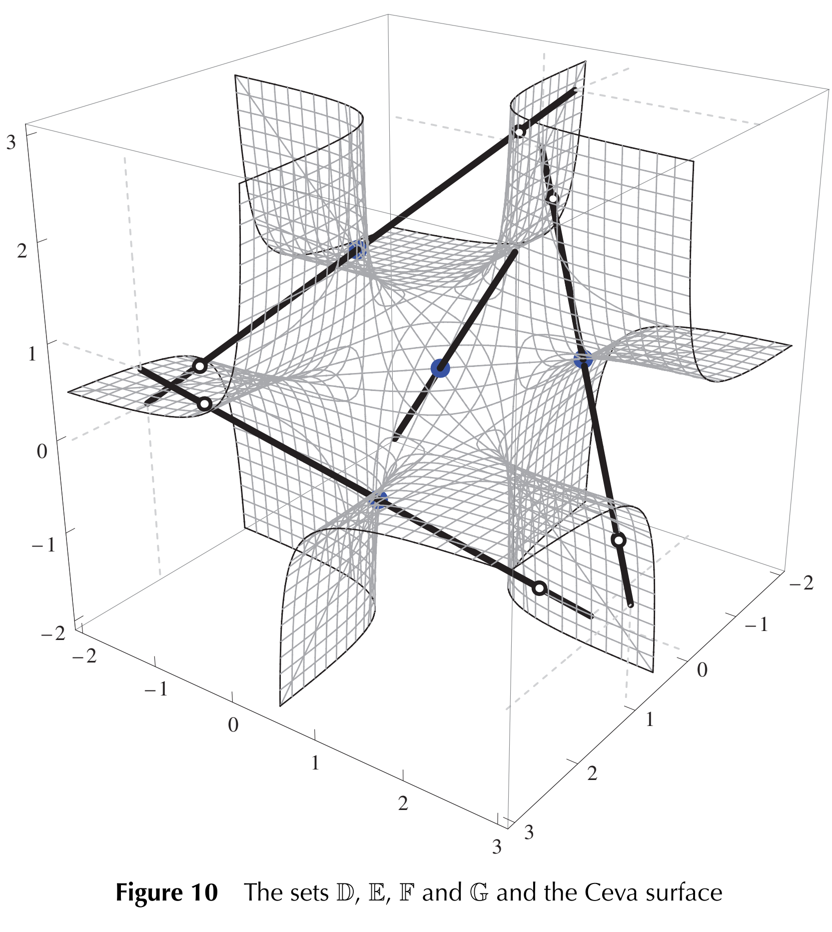

Figure 10 The sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{G}$ and the Ceva surface

yielding the "outer median triple" $(3/2, 1/2, -1/2)$. Intersecting $\mathbb{F}$ with the Ceva surface gives $(-1/2, 3/2, 1/2)$ and intersecting $\mathbb{G}$ with the Ceva surface results in $(1/2, -1/2, 3/2)$. Consequently, the only triples in $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ which correspond to concurrent cevians are the "median triple" and the three "outer median triples."

There is only a slight weakness in our argument above. In identifying the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$, we assumed that the triangles formed by the corresponding cevians have sides that are *parallel* to the cevians themselves. In [**3**], we proved that the only cevians AA_ρ, BB_σ, CC_τ that form triangles and with (ρ,σ,τ) not included in the sets $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$ are parallel cevians, that is the cevians AA_ρ, BB_σ, and CC_τ, where ρ, σ, τ satisfy (8) with the additional restriction

$$\xi \in \left(-\phi, -\phi^{-1}\right) \cup \left(\phi^{-2}, \phi^{-1}\right) \cup \left(\phi, \phi^{2}\right).$$

As it turns out, the properties analogous to (C) and (D) do not hold for triangles formed by such cevians. In conclusion, indeed, along with the medians and the median triangle, the outer medians and their outer median triangles are unique in satisfying all four properties analogous to those from the beginning of our note.

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Summary We define the notions of outer medians and outer median triangles. We show that outer median triangles enjoy similar properties to that of the median triangle.

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—Lee Sallows

ACROSS

1. Fake fight
5. Santa ____ (hot California winds)
9. Sight-related
14. Rice-shaped pasta
15. How Julius Caesar would write 2040
16. Marx brother
17. Takes an exam, in England
18. Number like 3 and 21
20. "Don't make _____!"
22. Thread holder
23. Number like 1.2904
26. Number like 55-Across
30. Salt Lake City native
32. A letter of the Greek alphabet
33. Neighborhood where you'd find a bodega
35. "Lost" creator J.J.
38. Number like 36
40. Treat with element #53, in Britain
41. Stories in une maison
42. Special effects in blockbusters: Abbr.
43. Digital book files
44. Number like two of the third roots of unity
48. Number like 12345
53. Large animal that represented the Egyptian god Set
55. A letter of the Greek alphabet
56. Number like e
61. Grp. for those over 50
62. Long, drawn-out attack
63. French town almost completely destroyed in the Battle of Normandy (partial abbr.)
64. No-____ condition: assumption that a viscous fluid has zero velocity relative to the boundary
65. More puzzling (not more like the number 7)
66. Number like 8842
67. Nine-digit IDs

DOWN

1. "Too bad, _____ . . ."
2. Number like 5
3. Central American civilization that used a base 20 number system
4. "Arrested Development" actress Portia de _____, or Manhattan Project physicist Bruno
5. Qty.
6. Spectroscopy method commonly used to gain information about the structure of organic molecules: Abbr.
7. There's one for x, y, and z
8. Apply quickly, like a sticker
9. "This is not good!"
10. 1945 Nobelist in Physics
11. Former MTV show hosted by Carson Daly
12. Typically very hoppy beer
13. It follows a thm.
19. Word that pairs with "neither"
21. Papa, Brainy, Harmony, and Handy, e.g.
24. Up _____ (stuck)
25. Of the flock, not the clergy
27. Approaches
28. "Don't look ____ like that!"
29. ___ Vegas
31. Egyptian for "be at peace", part of the name of a famous Egyptian chancellor and high priest
33. Darken
34. Current NPR White House correspondent Shapiro
35. Port city in Jordan that will be home to the world's only Star Trek-themed park
36. Pesters into doing, as in a task
37. Singer Corinne Bailey ___ or Carly ___ Jepsen
38. Stick that uses a spring
39. Star Trek phrase: "Set phasers to ____!"
40. Intl. justice group created in 2002 and headquartered in The Hague
43. Reveal
45. Antibacterial virus
46. Volume of a cube with side length 10 centimeters
47. Prefix with -morphism
49. Amherst sch. where mathematician Marshall Stone taught from 1968 to 1980
50. Numbers like -7 and pi
51. Line from a Lewis Carroll book: "I've often seen a cat without _____," thought Alice; "but _____ without a cat!"
52. Northern Scandinavians
54. "As seen ____!"
56. Prefix with -morphism
57. Free (of)
58. You might make a graph's edge this color
59. Part of 12-Down
60. Actor Chaney or Chaney Jr.

Types Theory

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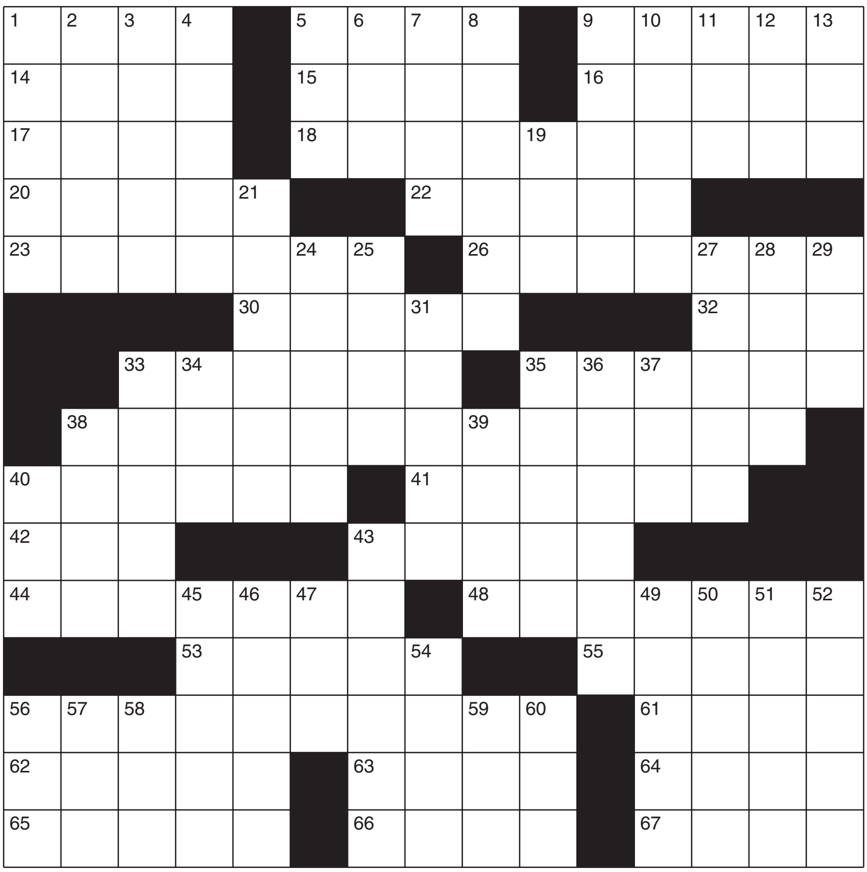

Clues are at left, on page 196. The solution is on page 211.

Extra copies of the puzzle, in both .pdf and .puz (AcrossLite) formats, can be found at the Magazine's website, or (temporarily) at http://www.mathematicsmagazine.org.

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Integer-Sided Triangles with Trisectible Angles

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Although every angle can be bisected using only a compass and unmarked straightedge, it is not possible to trisect every angle using only these tools. This is a well-known fact, with a 60° angle being the most common example of an angle that cannot be trisected. However, some angles (such as an angle of 45°) can be trisected (start with a 60° angle and bisect it twice). In this article, we pose the following problem: Can we find integer-sided triangles for which all three angles in the triangle are trisectible?

This problem has been solved by Chang and Gordon [**2**] for right triangles. It is proved there that the three angles in the right triangle with sides a, b, and c, where a, b, and c are relatively prime positive integers for which $a^2 + b^2 = c^2$, are trisectible if and only if c is a perfect cube. In this paper, we consider integer-sided triangles that do not have right angles. We show that there are infinitely many distinct (that is, not similar) integer-sided triangles with trisectible angles for which the triangles are scalene and do not have right angles. In particular, we provide explicit formulas for the sides of such triangles in Theorem 3 and Theorem 7. Since these formulas involve parameters with special properties, we give a number of examples to indicate how the parameters can be determined. We conclude by exploring some of the properties of these triangles and offering some suggestions for further study.

Consider a triangle with integer side lengths a, b, and c and with corresponding angles α, β, and γ. We express such triangles as (a, b, c) and, to avoid considering similar triangles, we focus our attention on triangles with $\gcd(a, b, c) = 1$. By the law of cosines, the numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are all rational; this fact is crucial to all that follows. For the angles α, β, and γ to be trisectible, the rational numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ must meet certain conditions.

Constructible angles and trisectible angles

Since the cosine values of the angles in our triangles are rational and the angles are required to be trisectible, we are led to the theory of constructible numbers.

Which numbers are constructible? A real number ρ is constructible if, given a line segment of length x, a line segment of length $|\rho|x$ can be constructed with a compass and unmarked straightedge. It is easy to show that rational numbers are constructible. With a little more effort, it can be shown that the set of constructible numbers is closed under the operations of addition, subtraction, multiplication, division, and extraction of square roots. In fact, a number ρ is constructible if and only if it can be obtained from rational numbers by a finite sequence of additions, subtractions, multiplications, divisions, and square roots. In particular, if ρ is a root of a quadratic polynomial with rational coefficients, then ρ is constructible.

Suppose that ρ is a root of a cubic polynomial C with rational coefficients. It turns out that ρ is constructible if and only if C has a rational root. For example, the number $\sqrt[3]{2}$ is not constructible since the polynomial $x^3 - 2$ has no rational roots. The

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reader can consult [**1**] or [**5**] for two of many possible sources for a discussion of these results. In addition, the geometry text by Isaacs [**7**] does a nice job presenting an informal discussion of constructible numbers and explaining why a 60° angle cannot be trisected.

Which angles are constructible? To avoid clutter, any reference to constructible or trisectible in this paper refers to constructions using only a compass and unmarked straightedge. Using basic right triangle trigonometry, there is a simple connection between constructible numbers and constructible angles. We leave the elementary proof of the following result to the reader.

THEOREM 1. *An angle θ is constructible if and only if $\cos\theta$ is a constructible number. In particular, if $\cos\theta$ is a rational number, then θ is constructible.*

There is nothing special about $\cos\theta$ in this theorem; we could just have easily used $\sin\theta$ or $\tan\theta$. We have focused on rational cosine values since we are interested in triangles with integer sides.

Which angles are trisectible? As we have mentioned, some angles (such as 20°) cannot be constructed, and thus some angles (such as 60°) cannot be trisected. When we say that an angle θ is trisectible, we mean that both θ and $\theta/3$ are constructible angles or (equivalently) both $\cos\theta$ and $\cos(\theta/3)$ are constructible numbers. It follows that the sum and difference of two trisectible angles is a trisectible angle, that the sum of a trisectible angle and a nontrisectible angle is a nontrisectible angle, and that any integer multiple of a trisectible angle is a trisectible angle. Using the properties of constructible numbers, we obtain the following result which is essential for our work. (For related results, the reader may find the paper by Chew [**3**] accessible and interesting.)

THEOREM 2. *Suppose that θ is an angle such that $\cos\theta$ is a rational number. The angle θ can be trisected if and only if there exists a rational number $t \in (-1, 1)$ such that $\cos\theta = 4t^3 - 3t$.*

Proof. Let θ be an angle such that $\cos\theta$ is a rational number. By Theorem 1, the angle θ can be constructed. The triple angle formula for cosine can be written as $\cos\theta = 4\cos^3(\theta/3) - 3\cos(\theta/3)$, indicating that $\cos(\theta/3)$ is a root of the cubic polynomial $4x^3 - 3x - \cos\theta$. By Theorem 1 and the fact that a root of a cubic polynomial with rational coefficients is constructible if and only if the cubic polynomial has a rational root, we obtain the following set of equivalences:

the angle θ is trisectible

$\Leftrightarrow$ the angle $\theta/3$ is constructible

$\Leftrightarrow$ $\cos(\theta/3)$ is a constructible number

$\Leftrightarrow$ the polynomial $4x^3 - 3x - \cos\theta$ has a rational root

$\Leftrightarrow$ there is a rational number $t \in (-1, 1)$ such that $\cos\theta = 4t^3 - 3t$.

The fact that t may be chosen in $(-1, 1)$ follows from the facts $f(-1) = f(1/2)$ and $f(1) = f(-1/2)$, where f is the function defined by $f(x) = 4x^3 - 3x$. ■

If two of the angles in a triangle are trisectible, then all three of the angles in the triangle are trisectible. For if the angles of the triangle are α, β, and γ, and the angles

α and β can be trisected, then we can construct the angle

$$\frac{\gamma}{3} = \frac{\pi}{3} - \frac{\alpha}{3} - \frac{\beta}{3},$$

showing that the third angle γ can be trisected. Note also that an angle θ and its supplement $\pi - \theta$ are either both trisectible or both nontrisectible. It follows that θ is trisectible if $|\cos\theta| = |4t^3 - 3t|$ for some rational number $t \in (0, 1)$.

Theorem 2 allows us to list all of the positive rational values for $|\cos\theta|$ when θ is trisectible. Suppose that y and z are relatively prime positive integers that satisfy $1 \le y < z$. Then the angle θ that satisfies

$$|\cos\theta| = \left|4\left(\frac{y}{z}\right)^3 - 3\left(\frac{y}{z}\right)\right| = \frac{y|4y^2 - 3z^2|}{z^3}$$

can be trisected. It is helpful to write this last fraction so that the numerator and denominator are relatively prime. There are three cases to consider, namely, z is odd, $z = 2w$ with w odd, and $z = 2w$ with w even. If z is odd, then the integers $y(4y^2 - 3z^2)$ and z^3 are relatively prime. To see this, suppose that p is a prime that divides both $y(4y^2 - 3z^2)$ and z^3. Then p is odd and p divides both z and $4y^2 - 3z^2$. But this implies that p divides $4y^2$ and thus y, a contradiction to the fact that y and z are relatively prime. Now suppose that $z = 2w$ with w odd. Then y is necessarily odd and

$$\frac{y(4y^2 - 3z^2)}{z^3} = \frac{y(4y^2 - 12w^2)}{8w^3} = \frac{y(y^2 - 3w^2)}{2w^3} = \frac{y(y^2 - 3w^2)/2}{w^3}.$$

Using an argument similar to the one just given, the numerator and denominator of this last fraction are relatively prime. Finally, suppose that $z = 2w$ with w even and write

$$\frac{y(4y^2 - 3z^2)}{z^3} = \frac{y(4y^2 - 12w^2)}{8w^3} = \frac{y(y^2 - 3w^2)}{2w^3}.$$

With similar reasoning, we find that the numerator and denominator of this last fraction are relatively prime.

In summary, an acute angle θ is trisectible if and only if $\cos\theta = m/n$, where m and n are relatively prime and are constructed in one of these three ways:

$m = y|4y^2 - 3z^2|$, $n = z^3$, with z odd, $\gcd(y, z) = 1$, and $1 \le y < z$;

$m = \frac{1}{2}y|y^2 - 3w^2|$, $n = w^3$, with w and y odd, $\gcd(y, w) = 1$, and $1 \le y < 2w$;

$m = y|y^2 - 3w^2|$, $n = 2w^3$, with w even, $\gcd(y, w) = 1$, and $1 \le y < 2w$.

Note that the denominator n is either an odd cube or twice an even cube. It is thus possible to list all of the possible positive rational values for $|\cos\theta|$ when θ can be trisected. The first few of these values (in the order of increasing denominators) are listed below:

$$1, \frac{9}{16}, \frac{11}{16}, \frac{5}{27}, \frac{13}{27}, \frac{22}{27}, \frac{23}{27}, \frac{27}{125}, \frac{37}{125}, \frac{44}{125}, \frac{71}{125}, \frac{91}{125}, \frac{99}{125},$$

$$\frac{117}{125}, \frac{118}{125}, \frac{7}{128}, \frac{47}{128}, \frac{115}{128}, \frac{117}{128}, \frac{18}{343}, \frac{73}{343}, \frac{143}{343}, \frac{207}{343}, \frac{235}{343},$$

$$\frac{262}{343}, \frac{297}{343}, \frac{305}{343}, \frac{332}{343}, \frac{333}{343}, \frac{107}{432}, \frac{143}{432}, \frac{413}{432}, \frac{415}{432}, \ldots$$

To find an integer-sided triangle (a, b, c) with three trisectible angles α, β, and γ, we need to choose the sides a, b, and c so that all three of the numbers $|\cos\alpha|$, $|\cos\beta|$, and $|\cos\gamma|$ appear somewhere in the extended list of possible values (along with the value 0 if the triangle is a right triangle).

Integer-sided triangles with trisectible angles

We will use the abbreviation ISTTA for an Integer-Sided Triangle with Trisectible Angles and use the term primitive ISTTA when the sides of the triangle are relatively prime.

Right triangles To find a primitive ISTTA with a right angle, the hypotenuse must be the cube of an odd number whose only prime factors are of the form $4k + 1$ (see Chang and Gordon [**2**]). The first such example that appears using the cosine values in our list is (44, 117, 125), which uses the prime 5. Two further examples are (828, 2035, 2197) (using the prime 13) and (495, 4888, 4913) (using the prime 17). A search for an ISTTA without a right angle was the motivation behind the results presented in this paper.

Isosceles triangles Since an angle of 60° is the classic example of an angle that cannot be trisected, we know that there does not exist an equilateral ISTTA. Since $\sqrt{2}$ is irrational, there are no integer-sided triangles that are both right and isosceles. Referring to the above list of cosine values, we can easily find primitive ISTTAs that are isosceles. If m/n is a rational number from our list (with $\gcd(m, n) = 1$), then the triangle $(n, n, 2m)$ is an integer-sided triangle for which the equal base angles are trisectible, and thus the triangle has three trisectible angles. If n is odd, the sides n and $2m$ have no common factors, while if n is even, we can factor a 2 (and 2 only) out of each term and obtain the triangle $(n/2, n/2, m)$ for which the sides have no common factors. For example,

$$\frac{9}{16} \to (8, 8, 9), \quad \frac{11}{16} \to (8, 8, 11), \quad \frac{5}{27} \to (27, 27, 10), \quad \frac{13}{27} \to (27, 27, 26),$$

and so on. A glance at the above list of cosine values reveals that the isosceles triangle of this type with the smallest perimeter is (8, 8, 9). By referring to our general formulas for m and n, we see that the two equal sides of a primitive ISTTA that is isosceles must be perfect cubes.

After seeing examples of right triangles and isosceles triangles that are ISTTAs, we are left wondering whether we can find a scalene ISTTA that does not have a right angle. At first glance, it may seem unlikely that the law of cosines can give an allowed cosine value for each of the three angles. But as we will see, such triangles do exist. In the next section, we present a simple approach for finding some of them, and later on we will consider a more general approach.

A simple approach for finding an ISTTA

In this section, we consider triangles with angles θ and 2θ, and triangles with angles $\pi - \theta$ and $\pi - 2\theta$.

Suppose that θ is a trisectible angle that satisfies $0 < \theta < \pi/3$ and $\cos\theta = m/n$, where m and n are relatively prime positive integers. Consider the triangle with angles θ, 2θ, and $\pi - 3\theta$. With a little thought, we see that such a triangle can be constructed and that all three of its angles are trisectible. Furthermore, the double and triple angle identities for cosine reveal that the cosine of each of the angles is a rational number. Since $\cos\theta$ is rational, the angle θ cannot assume any of the values $\pi/6$, $\pi/5$, or $\pi/4$. (The value of $\cos(\pi/5)$ is $\left(1+\sqrt{5}\right)/4$, which is half the golden mean.) It follows that the corresponding triangle is neither right nor isosceles. Letting a, b, and c denote the sides opposite angles θ, 2θ, and $\pi - 3\theta$, respectively, the law of sines yields

$$\frac{\sin\theta}{a} = \frac{\sin 2\theta}{b} = \frac{\sin(\pi - 3\theta)}{c} = \frac{\sin 3\theta}{c}.$$

Making the convenient choice $a = n^2$, we find that

$$\begin{aligned} b &= \frac{\sin 2\theta}{\sin\theta}\cdot a = 2a\cos\theta = 2mn; \\ c &= \frac{\sin 3\theta}{\sin\theta}\cdot a = \frac{\sin\theta\cos 2\theta + \sin 2\theta\cos\theta}{\sin\theta}\cdot a \\ &= (2\cos^2\theta - 1 + 2\cos^2\theta)\, a = 4m^2 - n^2. \end{aligned}$$

Note that c is positive since the conditions on θ imply that $m/n > 1/2$. The reader can verify that $\gcd(a, c) = 1$ when n is odd and $\gcd(a, c) = 4$ when n is even. It follows that $(n^2, 2mn, 4m^2 - n^2)$ is a primitive ISTTA when n is odd and that $(n^2/4, mn/2, (4m^2 - n^2)/4)$ is a primitive ISTTA when n is even.

Now suppose that θ is trisectible and satisfies $\pi/3 < \theta < \pi/2$ and $\cos\theta = m/n$, where m and n are relatively prime positive integers. Consider the triangle with angles $\pi - \theta, \pi - 2\theta$, and $3\theta - \pi$. As before, such a triangle can be constructed, all three of its angles are trisectible, and the cosine of each angle is a rational number. Since the corresponding triangle has an obtuse angle, it is certainly not a right triangle. Since $\cos(2\pi/5)$ is not rational, the angle θ cannot assume the value $2\pi/5$ and this shows that the corresponding triangle is not isosceles. Proceeding as above (with the details left to the reader), we find that the form of the triangle is now $(n^2, 2mn, n^2 - 4m^2)$ give or take a multiple of four. We have thus established the following result.

THEOREM 3. *If m and n are relatively prime positive integers and m/n is the cosine of a trisectible angle, then $(n^2, 2mn, |4m^2 - n^2|)$ is a primitive ISTTA when n is odd and $(n^2/4, mn/2, |4m^2 - n^2|/4)$ is a primitive ISTTA when n is even. Each of these triangles is scalene and does not have a right angle.*

It is easy to verify that the side n^2 or $n^2/4$ is always a sixth power of an integer. To illustrate this formula, note that

$$\frac{9}{16} \to (64, 72, 17), \qquad \frac{11}{16} \to (64, 88, 57),$$
$$\frac{5}{27} \to (729, 270, 629), \quad \text{and} \quad \frac{23}{27} \to (729, 1242, 1387).$$

It can be shown that different angles θ and ϕ yield different (that is, noncongruent) triangles. The details are elementary but a little tedious. (As a suggestion for the proof, show that the assumption that two different angles give the same triangle leads to an angle that has an irrational cosine value.) There are infinitely many distinct examples of these types of triangles.

This method can be extended to other angle combinations. For example, we can consider the triangle with angles $\theta, 3\theta, \pi - 4\theta$ under the assumption that $0 < \theta < \pi/4$ and the triangles with angles $\theta, 4\theta, \pi - 5\theta$ and $2\theta, 3\theta, \pi - 5\theta$, respectively, under the assumption that $0 < \theta < \pi/5$. Since the multiple angle identities for cosine become more involved as the multiple increases, the formulas for the sides of the triangles become more complicated as well.

Cosine values in integer-sided triangles

The method just described is elementary but (at least for the simple cases considered above) it limits the types of triangles that appear since the angles are all essentially multiples of a common angle and all of the cosine values have denominators consisting of the same number raised to different powers. What happens if we take two allowed cosine values with distinctly different denominators? To answer this question, we first look at the properties of the cosines of the angles of an integer-sided triangle.

Suppose that angles θ and ϕ have rational cosines that satisfy $0 < \theta < \phi < \pi/2$. Writing $\cos\theta = m/n$ and $\cos\phi = p/q$, we can compute the cosine of the remaining angle in the triangle:

$$\cos\big(\pi - (\theta + \phi)\big) = -\cos(\theta + \phi) = \sin\theta\sin\phi - \cos\theta\cos\phi$$
$$= \frac{\sqrt{n^2 - m^2}\sqrt{q^2 - p^2} - mp}{nq}.$$

In order for this number to be rational (which it must be for an integer-sided triangle), we need the integer $(n^2 - m^2)(q^2 - p^2)$ to be a perfect square. Each of these differences can be expressed as the product of a perfect square and a square-free number, say $n^2 - m^2 = j^2 r$ and $q^2 - p^2 = k^2 s$, where r and s are square-free. (A positive integer is said to be square-free if it is not divisible by any perfect square other than 1. We adopt the standard convention that 1 is square-free.) In order for the cosine of the third angle to be rational, it is necessary that $r = s$. This discussion leads to the following definition and related results.

DEFINITION. Let $\theta \in (0, \pi)$ be an angle whose cosine is rational and write $\cos\theta = m/n$, where m and n are integers. Express the number $n^2 - m^2$ as $j^2 r$, where j and r are positive integers with r square-free. We call the number r the *residual* of the angle θ.

If $\cos\theta$ is rational (where $\theta \in (0, \pi)$) and the residual of θ is r, then we can calculate that $\sin\theta = w\sqrt{r}$, where w is a positive rational number. Conversely, if $\cos\theta$ is rational and $\sin\theta = w\sqrt{r}$ (where r is a square-free integer) for some positive rational number w, then the residual of θ is r.

THEOREM 4. *Suppose that θ and ϕ are angles with rational cosines and common residual r. Then all of the angles $\pi - \theta$, $\theta + \phi$, 2θ, 3θ, and $\pi - \theta - \phi$ have rational cosines and residual r.*

Proof. We may assume that each of these angles lies in the interval $(0, \pi)$. Using basic trigonometric identities, it is easy to verify that the cosine of each of the listed angles is rational. By hypothesis, there exist integers m, n, p, and q such that $\cos\theta = m/n$ and $\cos\phi = p/q$, and there exist positive rational numbers w_1 and w_2 such that $\sin\theta = w_1\sqrt{r}$ and $\sin\phi = w_2\sqrt{r}$. Since $\cos(\pi - \theta) = -\cos\theta$ and

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta = \frac{p}{q}\cdot w_1\sqrt{r} + \frac{m}{n}\cdot w_2\sqrt{r} = w_3\sqrt{r}\,,$$

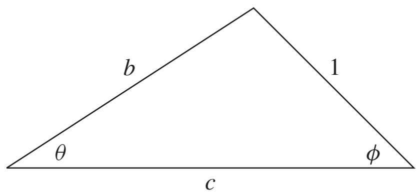
where w_3 is a positive rational number, the residual of the angles $\pi - \theta$ and $\theta + \phi$ is r. It follows easily that the angles $2\theta = \theta + \theta$, $3\theta = 2\theta + \theta$, and $\pi - \theta - \phi = \pi - (\theta + \phi)$ have residual r. ■

THEOREM 5. *If θ, ϕ, and ψ are the three angles of an integer-sided triangle, then the angles θ, ϕ, and ψ have the same residual.*

Proof. By relabeling the angles if necessary, we may assume that θ and ϕ are acute angles. Since the numbers $\cos\theta$, $\cos\phi$, and $\cos\psi$ are rational, the discussion before the definition of "residual" shows that the angles θ and ϕ have the same residual r. By Theorem 4, the angle $\psi = \pi - (\theta + \psi)$ also has residual r. ■

THEOREM 6. *Suppose that θ and ϕ are angles with rational cosines that satisfy $0 < \theta < \phi < \pi/2$. If θ and ϕ have the same residual, then there exists an integer-sided triangle containing the angles θ and ϕ.*

Proof. Suppose that the angles θ and ϕ have the same residual. We can then write $\cos\theta = m/n$, $n^2 - m^2 = j^2 r$, $\cos\phi = p/q$, and $q^2 - p^2 = k^2 r$, where all of the integers are positive and r is square-free. Consider the triangle with angles θ and ϕ and with sides as indicated in the figure.



Using the law of sines, we find that

$$\frac{\sin\theta}{1} = \frac{\sin\phi}{b} = \frac{\sin(\phi+\theta)}{c}.$$

It follows that

$$b = \frac{\sin\phi}{\sin\theta} = \frac{n\sqrt{q^2-p^2}}{q\sqrt{n^2-m^2}} = \frac{nk}{qj};$$

$$c = \frac{\sin(\phi+\theta)}{\sin\theta} = \frac{n}{\sqrt{n^2-m^2}}\left(\frac{m\sqrt{q^2-p^2}}{nq} + \frac{p\sqrt{n^2-m^2}}{nq}\right) = \frac{mk+pj}{qj}.$$

Hence, the triangle $\big(qj,\ nk,\ mk+pj\big)$ is an integer-sided triangle containing the angles θ and ϕ. ■

Theorems 5 and 6 together show that acute angles θ and ϕ with rational cosines have the same residual if and only if there exists an integer-sided triangle containing the angles θ and ϕ.

A general formula for an ISTTA

Combining Theorem 6 with our knowledge of the cosine values needed for trisectible angles, we obtain the following result. This theorem provides a general answer to the question posed at the beginning of the paper.

THEOREM 7. *Let θ and ϕ be trisectible angles with rational cosines that satisfy $0 < \theta < \phi < \pi/2$. Suppose that θ and ϕ have the same residual r and write $\cos\theta = m/n$, $n^2 - m^2 = j^2 r$, $\cos\phi = p/q$, and $q^2 - p^2 = k^2 r$, where all of the integers are positive. Then each of the triangles $\big(qj,\ nk,\ mk+pj\big)$ and $\big(qj,\ nk,\ mk-pj\big)$ is an ISTTA.*

Proof. The formula $\big(qj,\ nk,\ mk + pj\big)$ was determined in the proof of Theorem 6. By considering the acute angles θ and $\phi - \theta$ and using the same approach, we obtain the formula $\big(qj,\ nk,\ mk - pj\big)$. Note that this second triangle contains an obtuse angle. ■

Armed with the knowledge that the angles θ and ϕ need to have the same residual, we return to the list of possible rational cosine values for trisectible angles and compute the residual of each one. Once we find two values with the same residual, we can use the formulas $\big(qj,\ nk,\ mk \pm pj\big)$ to obtain the corresponding triangles. Doing so, we find that the angles associated with $5/27$ and $37/125$ both have residual 11, and thus obtain the triangles

$$(972, 1000, 476) = 4 \cdot (243, 250, 119) \quad \text{and} \quad (972, 1000, 116) = 4 \cdot (243, 250, 29).$$

We next find that the angles corresponding to $9/16$ and $47/128$ both have residual 7 and obtain the triangles

$$(640, 720, 640) = 80 \cdot (8, 9, 8) \quad \text{and} \quad (640, 720, 170) = 10 \cdot (64, 72, 17).$$

We also find that both $44/125$ and $117/125$ yield perfect squares (the residual is 1) and thus end up with the triangles

$$(5500, 14625, 15625) = 125 \cdot (44, 117, 125) \quad \text{and} \quad (5500, 14625, 11753).$$

The triangle (44, 117, 125) is the right triangle that was mentioned earlier. The triangle (5500, 14625, 11753), which has an obtuse angle, indicates that the common residual of the angles can be 1 even when the triangle is not a right triangle. Glancing at the six triangles just found, we realize that this method sometimes generates isosceles triangles or right triangles. One way to avoid these triangles is to start with two cosine values that have relatively prime denominators.

As we see in these examples, a scale factor may be needed to reduce the triangles to ones in which the sides are relatively prime. Assuming that $0 < \theta < \phi < \pi/2$ and adopting the notation of Theorem 7, suppose that $\gcd(m, n) = 1$, $\gcd(p, q) = 1$, and $\gcd(n, q) = 1$. Let $g = \gcd(j, k)$ and write $j = gj'$ and $k = gk'$. It is then an elementary task to prove that each of the formulas $\big(qj',\ nk',\ mk' + pj'\big)$ and $\big(qj',\ nk',\ mk' - pj'\big)$ yields a primitive ISTTA.

We now have two methods for finding ISTTAs. Although it is very unlikely, are we certain that the two methods are actually different? The first (and simpler) method presented in Theorem 3 resulted in triangles in which the angles are rational multiples of each other modulo π. Suppose that θ, ϕ, and ψ are the three angles in an ISSTA. Are there integers n, r, and s such that either $\theta = n\pi + (r/s)\phi$ or $\theta = n\pi + (r/s)\psi$? Some elementary algebra yields $s\theta = ns\pi + r\phi$ for the first equation and

$$s\theta = ns\pi + r(\pi - \theta - \phi) \quad \text{or} \quad (r + s)\theta = (ns + r)\pi - r\phi$$

for the second equation. In either case, we can find positive integers r and s such that $\cos(s\theta) = \pm\cos(r\phi)$.

The Chebyshev polynomials T_k are useful at this stage of the process. One of the many properties of these polynomials is the fact that $\cos(k\theta) = T_k(\cos\theta)$ for each positive integer k. In addition, the polynomial T_k has degree k with leading coefficient

2^{k-1}. If we assume that $\cos\theta = m/n$ and $\cos\phi = p/q$, where n and q are relatively prime odd numbers and $\gcd(m, n) = 1 = \gcd(p, q)$, then we find that

$$\cos(s\theta) = T_s(\cos\theta) = 2^{s-1}\left(\frac{m}{n}\right)^s + \cdots = \frac{2^{s-1}m^s + nX}{n^s},$$
$$\cos(r\theta) = T_r(\cos\theta) = 2^{r-1}\left(\frac{p}{q}\right)^r + \cdots = \frac{2^{r-1}p^r + qY}{q^r},$$

where X and Y are integers. Since these two quantities are equal in magnitude, we find that

$$q^r\left(2^{s-1}m^s + nX\right) = \pm n^s\left(2^{r-1}p^r + qY\right),$$

revealing that one of the odd primes that divides q must also divide $2^{r-1}p^r$, a contradiction. It follows that the method of Theorem 7 generates triangles that do not appear using the method of Theorem 3.

A little number theory

To obtain these types of triangles efficiently, we need a better method than random searching for finding common residuals. Recall that our list of rational cosine values can be generated by numbers of the form $|4t^3 - 3t|$, where t is a rational number in the interval $(-1, 1)$. As the following lemma shows, the angles associated with t and $|4t^3 - 3t|$ have the same residual since the angles associated with $4t^3 - 3t$ and $-(4t^3 - 3t)$ are supplements of each other.

LEMMA. *Suppose that t is a rational number in the interval $(-1, 1)$. If angles θ and ϕ in $(0, \pi)$ satisfy $\cos\theta = t$ and $\cos\phi = 4t^3 - 3t$, then θ and ϕ have the same residual.*

Proof. By the triple angle formula for cosine, we find that $\cos(3\theta) = \cos\phi$. It follows that the angle 3θ is either ϕ, $2\pi - \phi$, or $2\pi + \phi$. Hence, the angles ϕ and 3θ have the same residual. Referring to Theorem 4, we see that the angles θ and ϕ have the same residual. ∎

Let r be a square-free number and, to simplify the notation, let f be defined by $f(t) = 4t^3 - 3t$. Given any two positive integers x and y, the rational number $|x^2 - ry^2|/(x^2 + ry^2)$ lies in the interval $(0, 1)$. Let θ be the acute angle for which $\cos\theta$ has this value. Then it is easy to verify that r is the residual of θ and, by Theorem 2 and the above lemma, the acute angle ϕ that satisfies

$$\cos\phi = \left|f\left(\frac{x^2 - ry^2}{x^2 + ry^2}\right)\right|$$

is trisectible and has residual r. For a given value of r, we are thus interested in numbers that can be expressed in the form $x^2 + ry^2$. Going a step further, we can ask which prime numbers p can be expressed in this way. It is known that the equation $p = x^2 + ry^2$ has an infinite number of solutions when r has any of the values 1, 2, 3, or 7 and that in each case, primes of a certain form are needed. In particular, the theory of algebraic integers (see Hardy and Wright [**6**] or Niven and Zuckerman [**8**]) indicates that the equation can be solved in the following cases:

$r = 1$ corresponds to primes of the form $4k + 1$;

$r = 2$ corresponds to primes of the form $8k + 1$ or $8k + 3$;

$r = 3$ corresponds to primes of the form $3k + 1$;

$r = 7$ corresponds to odd primes of the form $7k + 1$, $7k + 2$, or $7k + 4$.

The first few examples of each type are as follows.

$5 = 1^2 + 2^2$	$3 = 1^2 + 2 \cdot 1^2$	$7 = 2^2 + 3 \cdot 1^2$	$11 = 2^2 + 7 \cdot 1^2$
$13 = 2^2 + 3^2$	$11 = 3^2 + 2 \cdot 1^2$	$13 = 1^2 + 3 \cdot 2^2$	$23 = 4^2 + 7 \cdot 1^2$
$17 = 1^2 + 4^2$	$17 = 3^2 + 2 \cdot 2^2$	$19 = 4^2 + 3 \cdot 1^2$	$29 = 1^2 + 7 \cdot 2^2$
$29 = 2^2 + 5^2$	$19 = 1^2 + 2 \cdot 3^2$	$31 = 2^2 + 3 \cdot 3^2$	$37 = 3^2 + 7 \cdot 2^2$
$37 = 1^2 + 6^2$	$41 = 3^2 + 2 \cdot 4^2$	$37 = 5^2 + 3 \cdot 2^2$	$43 = 6^2 + 7 \cdot 1^2$
$41 = 4^2 + 5^2$	$43 = 5^2 + 2 \cdot 3^2$	$43 = 4^2 + 3 \cdot 3^2$	$53 = 5^2 + 7 \cdot 2^2$

To illustrate how these equations can be used to generate integer-sided trisectible triangles, we consider the case in which $r = 2$. Using the expressions for 11 and 3 in the second column, we obtain angles θ and ϕ (with $\theta < \phi$ as used in Theorem 7) as follows:

$$\cos\theta = \frac{m}{n} = \left| f\left(\frac{9-2}{9+2}\right) \right| = \frac{1169}{1331} \approx 0.8783$$

and

$$\cos\phi = \frac{p}{q} = \left| f\left(\frac{1-2}{1+2}\right) \right| = \frac{23}{27} \approx 0.8519.$$

Using the formulas $(qj, nk, mk \pm pj)$ from Theorem 7, we obtain the two triangles (1215, 1331, 2204) and (1215, 1331, 134). The reader can use the above equations to obtain many more such triangles for r values of 1, 2, 3, or 7.

It is also interesting to investigate what happens for other values of r, including values of r that are not prime. For one example, let $r = 57$ and note that the cosine values $8/11$ and $28/29$ have residual 57. It follows that

$$f\left(\frac{8}{11}\right) = -\frac{856}{1331} \quad \text{and} \quad f\left(\frac{28}{29}\right) = \frac{17164}{24389}$$

and these values lead to the triangles (22627, 24389, 31716) and (22627, 24389, 2612). The reader interested in pursuing examples of this type may find the book by Cox [**4**] quite useful.

Counting primitive ISTTAs with a given residual

Given a square-free integer r, are we guaranteed that there exists an ISTTA for which the common residual of the angles is r? For $r = 1$, we know that there are an infinite number of distinct possibilities based on our earlier comments concerning right triangles. In fact, by using the acute angles of these right triangles and applying the second formula in Theorem 7 (which gives a triangle with an obtuse angle), we obtain an infinite number of scalene ISTTAs without a right angle whose angles have common residual 1. The following theorem shows that this result extends to all other values of r. The adjective "distinct," as used in the statement of the theorem, refers to triangles that are not similar. We continue to make use of the function f defined by $f(t) = 4t^3 - 3t$.

THEOREM 8. *For each square-free integer r, there exist an infinite number of distinct scalene ISTTAs for which the common residual of the angles is r.*

Proof. Suppose first that r is an odd square-free integer. For each positive integer n, consider the rational number $a_n = |4n^2 - r|/(4n^2 + r)$. Choose a positive integer N such that $N > 2\sqrt{r}$. The reader may verify that the sequence $\{a_n\}_{n=N}^{\infty}$ is increasing and that $\sqrt{3}/2 < a_n < 1$ for all $n \geq N$. Noting that

$$(4n^2 + r)^2 - (4n^2 - r)^2 = (4n)^2 r$$

and referring to the lemma, we find that the angle θ_n that satisfies $\cos\theta_n = f(a_n)$ is an acute angle that has residual r. Since the sequence $\{a_n\}_{n=N}^{\infty}$ is increasing to 1 and the function f is increasing on the interval $\left(\sqrt{3}/2, 1\right)$, the sequence $\{\theta_n\}_{n=N}^{\infty}$ is decreasing to 0. From this infinite collection of angles, we can take pairs of θ values and use them (as in Theorem 7) to form an infinite number of distinct ISTTAs whose angles have residual r. Using θ_q as our base angle, where $\theta_q < \pi/4$, the angles θ_q and θ_n for $n > q$ generate triangles that have two unequal acute angles and an obtuse angle. Therefore, there exist an infinite number of distinct scalene ISTTAs whose angles have residual r.

Now suppose that r is an even square-free integer. For each positive integer n, consider the rational number $b_n = (n^2 r - 1)/(n^2 r + 1)$. The reader may verify that the sequence $\{b_n\}$ is increasing and that $\sqrt{3}/2 < b_n < 1$ for all $n \geq 3$. Noting that

$$(n^2 r + 1)^2 - (n^2 r - 1)^2 = (2n)^2 r$$

and referring to the lemma, we find that the angle ϕ_n that satisfies $\cos\phi_n = f(b_n)$ is an acute angle that has residual r. The rest of the proof for this case now follows the lines of the previous paragraph. ■

This result is primarily of theoretical interest, since the sides of the triangles become very large even for small values of r. We leave such calculations to the reader.

Some further comments

Now that we have general methods for obtaining scalene ISTTAs that do not have a right angle, we can step back and ask various questions about them. For instance, which of these triangles has the least perimeter? Looking over the triangles we have found, the one with the smallest perimeter is (64, 72, 17), which has perimeter 153. Can we find a smaller one? It certainly appears that as the denominators of the cosine fractions increase so does the perimeter. But, as we have seen, there may be some values that generate numbers that involve a large scale factor and thus give a triangle with smaller sides. To be certain that this is the best we can do, we can write a program that checks all possible integer-sided triangles with perimeter less than 153 and determines if the corresponding angles are trisectible. There are various ways to write a program to do this and we leave the details to the reader. Our results show that 153 is indeed the smallest perimeter for a triangle of this type. The next smallest value for the perimeter is 209 from the triangle (64, 88, 57).

For a different type of question, suppose that θ, ϕ, and ψ are the angles of an ISTTA. Under what conditions are all three cosine values $\cos(\theta/3)$, $\cos(\phi/3)$, and $\cos(\psi/3)$ rational? Referring to Theorem 2, we know that $\{|f(t)| : t \in \mathbb{Q} \cap (-1, 1)\}$ (where $f(x) = 4x^3 - 3x$) is the set of possible rational cosine values for trisectible acute angles. Suppose, for instance, that $t = 1/3$. In this case, we find $f(t) = -23/27$ and thus the acute angle θ that satisfies $\cos\theta = |f(t)| = 23/27$ is trisectible. However, the rational root $-1/3$ of the polynomial $4x^3 - 3x - \cos\theta$ cannot be the positive number $\cos(\theta/3)$. Determining the other two roots of the polynomial and taking the positive root reveals that $\cos(\theta/3)$ is an irrational number, namely $\big(1+2\sqrt{6}\big)/6$. Hence, the numbers $\cos(\theta/3)$, $\cos(\phi/3)$, and $\cos(\psi/3)$ may not be rational even when θ, ϕ, and ψ are the trisectible angles of an integer-sided triangle. As indicated by our example, problems arise when $f(t) < 0$. If we restrict our cosine values to the set $\{f(t) : t \in \mathbb{Q} \cap (\sqrt{3}/2, 1)\}$, then the angle θ arising from one of these t values satisfies $\cos\theta = f(t)$ and $\cos(\theta/3) = t$. As indicated by the next theorem, a further condition is required to guarantee that all three values $\cos(\theta/3)$, $\cos(\phi/3)$, and $\cos(\psi/3)$ are rational.

THEOREM 9. *Suppose that θ, ϕ, and ψ are the angles of an integer-sided trisectible triangle. The three numbers* $\cos(\theta/3)$, $\cos(\phi/3)$, *and* $\cos(\psi/3)$ *are rational if and only if the common residual of the angles θ, ϕ, and ψ is* 3.

Proof. Suppose that the three numbers $\cos(\theta/3)$, $\cos(\phi/3)$, and $\cos(\psi/3)$ are rational. Let r be the common residual of the angles θ, ϕ, and ψ. By Theorem 4, the angles $\theta/3$, $\phi/3$, $(\theta+\phi)/3$, and $\psi/3$ also have residual r. Therefore, there exist positive rational numbers w_1, w_2, and w_3 such that

$$\cos(\psi/3) = w_1, \qquad \sin(\psi/3) = w_2\sqrt{r}, \qquad \sin\big((\theta+\phi)/3\big) = w_3\sqrt{r}.$$

Using these values and the addition formula for sine, we find that

$$\begin{aligned} w_3\sqrt{r} &= \sin\left(\frac{\theta+\phi}{3}\right) = \sin\left(\frac{\pi}{3} - \frac{\psi}{3}\right) = \frac{\sqrt{3}}{2}\cdot w_1 - \frac{1}{2}\cdot w_2\sqrt{r} \\ &\Leftrightarrow 2w_3 r = w_1\sqrt{3r} - w_2 r. \end{aligned}$$

Since $\sqrt{3r}$ is a rational number, we find that $r = 3$.

Now suppose that the common residual of the angles θ, ϕ, and ψ is 3. Since θ is a trisectible angle with a rational cosine, the polynomial $4x^3 - 3x - \cos\theta$ has a rational root, which we will denote by m/n. As the reader may verify, the roots of the polynomial $4x^3 - 3x - \cos\theta$ are thus

$$\frac{m}{n}, \; -\frac{m}{2n} \pm \frac{\sqrt{3(n^2-m^2)}}{2n}.$$

Choose an angle $\alpha \in (0, \pi)$ such that $\cos\alpha = m/n$. By the lemma, we find that the angles α and θ have the same residual, namely 3, and thus $n^2 - m^2 = 3j^2$ for some positive integer j. It follows that all three roots of the polynomial $4x^3 - 3x - \cos\theta$ are rational and this guarantees that the value of $\cos(\theta/3)$ is rational. The same reasoning shows that $\cos(\phi/3)$ and $\cos(\psi/3)$ are rational as well. ∎

To give an example of triangles of this type, let

$$\cos\left(\frac{\theta}{3}\right) = \frac{37}{38}, \quad \cos\theta = f\left(\frac{37}{38}\right) = \frac{5291}{6859}, \quad \text{and}$$
$$\cos\left(\frac{\phi}{3}\right) = \frac{13}{14}, \quad \cos\phi = f\left(\frac{13}{14}\right) = \frac{143}{343},$$

where we are assuming that both θ and ϕ are acute angles. These values for the angles generate the triangles (4802, 6859, 7293) and (4802, 6859, 3289). For these two cases, letting ψ denote the third angle, we find that

$$\cos\left(\frac{\psi}{3}\right) = \frac{241}{266}, \quad \cos\psi = \frac{604187}{2352637}, \quad \text{and}$$

$$\cos\left(\frac{\psi}{3}\right) = \frac{263}{266}, \quad \cos\psi = \frac{2117413}{2352637},$$

respectively.

Returning to the proof of the theorem, note that the three roots of the polynomial $4x^3 - 3x - \cos\theta$ can be written as

$$\cos\alpha, \ -\frac{1}{2}\cos\alpha \pm \frac{\sqrt{3}}{2}\sin\alpha \quad \text{or} \quad \cos\alpha, \ \cos\bigl(\alpha \pm 120^\circ\bigr),$$

where α is an angle for which $\cos\alpha = m/n$. For the general case in which the angle θ is trisectible, we know that one of these values is rational and that one of these values is $\cos(\theta/3)$. However, the theorem indicates that we are only guaranteed that $\cos(\theta/3)$ is the rational value for all of the angles when the common residual of the angles is 3.

Conclusion

The main goal of this paper has been to determine methods for finding integer-sided triangles with three trisectible angles. Several relatively simple methods can be found scattered throughout the paper. A number of other questions and problems concerning triangles with trisectible angles can be pursued. We briefly mention three of these.

i. We found that the ISTTAs with the least perimeter are (64, 72, 17) and (64, 88, 57). The common residual is 7 for the angles in the first triangle and 15 for the second one. It might prove interesting to determine the ISTTA with the least perimeter for other values of the residual r. We could also determine the number of ISTTAs with perimeter less than (say) 5000 and look for patterns in their residuals.

ii. Using ideas in this paper, it is not difficult to prove that the three altitudes (and thus the area) of an integer-sided triangle are rational if and only if the common residual of the three angles is 1. It might prove interesting to search for the triangles of these types (both with and without trisectible angles) that have the least perimeter, the least area, or the least altitude. Is one of the altitudes always an integer? Can all three altitudes be integers when the sides are relatively prime?

iii. Suppose that a is either a perfect cube or a perfect sixth power. How many distinct ISSTAs are there with a as one of the side lengths? How many of these are isosceles or right?

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Summary We consider the problem of finding integer-sided triangles for which all three angles in the triangle can be trisected with a compass and unmarked straightedge. Since some angles (such as 60°) cannot be trisected using only these tools, some care is required to find triangles with these properties. By the law of cosines, the cosines of the angles are rational numbers (since the sides of the triangles are integers). In order for the three

angles of the triangle to be trisectible, the rational cosine values must meet certain conditions. Using some elementary aspects of the theory of constructible numbers, we obtain several general methods for finding triangles that meet our conditions, then present some examples and explore a few properties of these triangles.

RUSSELL A. GORDON received his Ph.D. from the University of Illinois in 1987, writing his dissertation under the influence of Jerry Uhl. He has been teaching mathematics at Whitman College since then and is becoming increasingly aware that his current students believe that 1987 was a long time ago. Attending a Ke$ha concert with his teenage son while working on this paper helped convince his students that he is not completely ignorant of twenty-first century pop culture. When not pursuing various mathematical ideas, he enjoys eating his spouse's wonderful vegetarian cooking (for which doing the dishes is a small price to pay), watching movies with his family, and hiking in the local mountains.

Solution to puzzle on page 197:

1 S	2 P	3 A	4 R	■	5 A	6 N	7 A	8 S	■	9 O	10 P	11 T	12 I	13 C
14 O	R	Z	O	■	15 M	M	X	L	■	16 H	A	R	P	O
17 S	I	T	S	■	18 T	R	I	A	19 N	G	U	L	A	R
20 A	M	E	S	21 S	■	■	22 S	P	O	O	L	■	■	■
23 D	E	C	I	M	24 A	25 L	■	26 O	R	D	I	27 N	28 A	29 L
■	■	■	■	30 U	T	A	31 H	N	■	■	■	32 E	T	A
■	■	33 B	34 A	R	R	I	O	■	35 A	36 B	37 R	A	M	S
■	38 P	E	R	F	E	C	T	39 S	Q	U	A	R	E	■
40 I	O	D	I	S	E	■	41 E	T	A	G	E	S	■	■
42 C	G	I	■	■	■	43 E	P	U	B	S	■	■	■	■
44 C	O	M	45 P	46 L	47 E	X	■	48 N	A	T	49 U	50 R	51 A	52 L
■	■	■	53 H	I	P	P	54 O	■	■	55 O	M	E	G	A
56 I	57 R	58 R	A	T	I	O	N	59 A	60 L	■	61 A	A	R	P
62 S	I	E	G	E	■	63 S	T	L	O	■	64 S	L	I	P
65 O	D	D	E	R	■	66 E	V	E	N	■	67 S	S	N	S

Surprises

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This article is dedicated to the memory of Alla Bogomolnaya (1957–2012), an extraordinary teacher of mathematics and statistics.

Perhaps the most surprising thing about mathematics is that it is so surprising.
—E. C. Titchmarsh

In great mathematics there is a very high degree of unexpectedness...
—G. H. Hardy

My momma always said, "Life is like a box of chocolates. You never know what you're gonna get." —Forrest Gump

In my study and teaching of mathematics, surprises have always played an important part. Here I want to share some of these experiences with you. I feel a bit uneasy talking about them. Surprises are subjective, and others may not feel the same. It is like a famous joke about two people standing at the edge of a cliff. One looks around and says: "God... What beauty!" Another looks, and looks, and looks, and then asks: "Where?" The first does not know what to say or do. He just pushes the second off the cliff. Gently.

So, just in case, do not be too close to me while reading this.

My surprises are of various natures. Here I tried to organize them in four categories.

- How could I not know this for so long?
- Surprising, but not that much. Maybe, I could discover this myself...
- Even after I see a proof, the fact is still mind boggling. I could not discover this.
- How is it possible that this problem is not solved yet?

Let me now illustrate each category with examples and comments. Some of the examples are very elementary. I believe that the reader may enjoy spending a few minutes thinking about the problems. Solutions to most of the problems are omitted. Some of them can be found easily, and for others we provide references.

How could I not know this for so long?

A good thing about these is that we can blame them on our teachers.

Surprise 1. All parabolas are similar.

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COMMENTARY. A figure A in a plane is called *similar* to a figure B in the plane, if there exist a positive number k and a bijection $f : A \to B$ such that

$$\mathrm{dist}(f(x), f(y)) = k \cdot \mathrm{dist}(x, y)$$

for any $x, y \in A$. Any two segments are similar, any two circles are similar, any two equilateral triangles are similar. However, not every pair of ellipses are similar, and not every pair of hyperbolas are similar.

So why are all parabolas similar? Is it not true that the parabola with equation $y = x^2$ is wider than the one with equation $y = 2x^2$ and narrower than the one with equation $y = x^2/2$? No! The second is just smaller, and the third is larger, but all three are of the same shape, i.e., are similar. Though stretching or shrinking along the y-axis alone does not always transform a curve to a similar one, it does for parabolas.

The property is obvious from the definition of a parabola as a locus. For parabolas A and B, the constant k is just the ratio of distances from the foci to the directrices of the two parabolas.

I learned this in 2007, while working with O. Byer and D. L. Smeltzer on our book [**12**]. Later we discovered that, of course, the property was known. It turns out that Johannes Kepler mentioned it with great excitement in 1604! See [**21**].

Surprise 2. Suppose we have a perfectly spherical earth, density is distributed spherically symmetrically, and a cannonball is moving without drag under the influence of the gravitational field. Then the trajectory is a conic section. What is it?

COMMENTARY. It is . . . an ellipse, not a parabola as we are often taught.

I learned this from an article by L. M. Burko and R. H. Price [**11**]. It immediately made perfect sense to me. I had known for a long time that parabolic trajectory of a projectile is rare. However, for some strange reason, many texts stated that every thrown stone moved along a parabola. This also made perfect sense given the fact that the shape of a very narrow ellipse is close to a parabola: The latter can be considered as an ellipse with one focus removed to infinity. It was also surprising to see in the article a quotation from Newton's *Principia*, where he points out that Galileo's model, which assumes flat-earth-uniform gravitation, leads to a parabolic trajectory. But the central force model, which is used in astronomy, leads to an ellipse. I do not remember seeing this discussion in calculus texts.

Surprise 3. Characterize the set of all functions f that have continuous nth derivatives on an open interval $I \subset \mathbb{R}$ and can occur as a solution of some differential equation of the form

$$f^{(n)} + p_1 f^{(n-1)} + \cdots + p_{n-1} f' + p_n f = 0,$$

for some continuous functions $p_1, \ldots, p_n$ on I.

COMMENTARY. I asked this question when I taught an undergraduate course on differential equations. An exercise in the book by E. Boyce and R. C. DiPrima [**10**] asked to show that $f(x) = \sin(x^2)$ cannot be a solution of such an equation for $n = 2$. This can be easily seen from the theorem about the uniqueness of the solution of the initial value problem in this case. Indeed, the function that is identically zero on the interval is, obviously, a solution of this equation. As $f(0) = f'(0) = 0$, the function f would represent another solution with the same initial values, a contradiction.

Asking the question above was natural for a person with a background in algebra. It was surprising to me that this question was new to people who work with differential equations, and that it was not interesting to them.

I found the answer, but had difficulty proving it. It was my colleague David Bellamy who provided the first proof. The question appeared as a Problem in the *Monthly* [**5**]. Several people submitted much simpler solutions than ours; see [**6**].

Surprise 4. Let $\mathbb{Z}$ denote the ring of integers. Find all solutions $x \in \mathbb{Z}^n$ of a system of linear equations $Ax = b$, where A is a given $m \times n$ matrix over $\mathbb{Z}$, and $b \in \mathbb{Z}^m$ is a given vector.

COMMENTARY. The question appeared when I taught a graduate topics course on asymptotic design theory. It was very surprising to me that having taught algebra and linear algebra many times, and knowing about the structure of finitely generated modules over PIDs, I had not recognized the question. It turned out that my experience was not so unique: The related paper got accepted quickly in this MAGAZINE [**23**]. It was exactly this question that led H. J. S. Smith to his normal form for matrices [**28**].

Surprise 5. Find all real values of a such that the sequence $\{a_n\}_{n\geq 0}$ defined by $a_0 = a$, and $a_{n+1} = a_n^2 - 2$ for $n \geq 0$, converges.

COMMENTARY. I can think of at least three surprises related to this problem.

The first surprise was the unusual story behind the question. I assigned it as one of many other homework problems on limits in a high school where I was working at the time (1977). I had no idea that it was a challenge. I just thought that it was a nice extension to a simple question: What is the limit of $\{a_n\}_{n\geq 0}$ if it converges? After my students could not solve the problem, I tried it for two days, with no success. There was something very unusual about the sequence. I began asking my colleagues, and, soon, Yurii Pilipenko saw the light, and we finished a proof quickly. Several years later, we submitted it as a problem to the *Monthly* [**24**].

Another surprising thing about this sequence is that if it converges, then it must stabilize: All terms, starting at an arbitrary term, must be equal to its limit, which, obviously, can take one of two values: -1 or 2 (depending on a). This allows us to find a, and the set of all such a's allows a simple description. I had not seen anything like this before I asked the question. As I understood later, this was my first exposure to an interesting dynamical system and "repulsors." In 1985, Emil Grosswald pointed out to me that the set of values of a for which the sequence converges is uniformly dense on $[-1, 2]$.

The third surprise was when I saw solutions and extensive comments sent by readers of the *Monthly* [**25**]—eighty-two people from twenty-three countries! It turned out that this type of sequence had been studied long ago, since 1918 at least, and by many mathematicians. The corresponding area (which used to be called just analysis) is now called topological dynamics. Many references and generalizations were mentioned. Studies of iterations of the map $z \mapsto z^2 + c$ over $\mathbb{C}$, $c \in \mathbb{C}$, led to the notions of Fatou sets and Julia sets. Some people pointed out that it was a good example of how one can get a wrong answer by experimenting with a computer.

Surprise 6. Given two polygonal regions in a plane of the same area, one can be dissected by straight lines into finitely many smaller polygons such that the other can be assembled from them.

COMMENTARY. Sometimes this statement is called the Wallace-Bolyai-Gerwien Theorem, and proofs were found independently by W. Wallace, F. Bolyai, and P. Gerwien, who published them in 1831, 1833, and 1835, respectively. The polygonal regions do not have to be convex, or to have equal numbers of sides. I found it very surprising that many people to whom I mentioned this result did not know it. It is a little better known that in space (3-dimensional), the analog of the theorem does not hold: For example, one cannot dissect a cube by planes and assemble a regular tetrahedron from them. This follows from M. Dehn's solution of Hilbert's third problem in 1900. For details, extensions, and a proof of the Wallace-Bolyai-Gerwien Theorem, see V. G. Boltyanskii [**8**], or [**12**]. The proof of Dehn's theorem was simplified many times, and the version in [**8**] is one of the most beautiful proofs I have ever understood. Still, when it comes to surprises, the affirmative result in the plane is far ahead (for me) of the negative one in space.

Surprising, but not that much. Maybe, I could discover this myself...

For me, these surprises are the most numerous. Their frequency depends on how persistent I am in getting to the "bottom of things," on knowledge of related subjects, and on self-confidence.

Surprise 7. A watermelon is 99% water. One ton of watermelons was shipped, and during the shipment some water evaporated. The watermelons that arrived were made up of 98% water. What was the weight of the shipment when it arrived?

COMMENTARY. Well, solve it, as I did. The answer is $1/2$ ton. After this problem, my belief that I had good intuition about percents was shattered.

Surprise 8. It takes three days for a motorboat to travel from A to B down a river, and it takes it four days to come back. How long will it take a wooden log to be carried from A to B by the current?

COMMENTARY. This problem was one among twenty that my mathematics teacher, L. I. Bogomolny, assigned for the summer after the eighth grade. I spent a lot of time on it, unable to solve it. I was sure that some data was missing. My older brother Lazar solved it for me. I was amazed by the power of algebra, when he introduced more unknowns than he could find, and the answer appeared as the ratio of two of them. By the way, the answer is 24 days, and it is impossible to find the speed of the current, the speed of the boat, or the distance AB.

Surprise 9. Consider any positive integer N whose (decimal) digits read from left to right are in non-decreasing order, and whose last two digits (tens and ones) are in increasing order. Prove that the sum of the digits of $9N$ is always 9.

COMMENTARY. It was hard to believe, since N could be really large. For example, if $a = 1778$, $b = 2344459$, and $c = 12225557779$, then

$$9a = 16002,\ \ 9b = 21100131,\ \ 9c = 110030020011,$$

and the sum of digits in each case is 9. The proof is easy. If you find it for 3- or 4-digit numbers, the generalization is trivial. This problem was mentioned to me about ten years ago by V. A. Kanevsky.

Surprise 10. Take a four-digit number (in base 10) with not all digits equal. Rearranging its digits in decreasing order we get a number M. Rearranging its digits in increasing order, we get a number m. Consider $M - m$, and repeat the procedure. Do it again, and again .... After several iterations we get to the number 6174.

COMMENTARY. If this is not surprising, then I give up! I found a proof, but it was not illuminating (case analysis). A similar question can be asked for numbers with an arbitrary number of digits, and when bases different than 10 are used. Answers become more interesting. Play with a computer. See an article on Wikipedia (http://en.wikipedia.org/wiki/6174) for more information.

Surprise 11. $e^{i\pi} + 1 = 0$, or, more generally, $e^{ix} = \cos x + i \sin x$.

COMMENTARY. Please do not become angry with me for placing this result in this section ("...Maybe, I could discover this"). Yes, the equality $e^{i\pi} + 1 = 0$ is considered to be one of the highest standards of mathematical beauty: It ties five of the most celebrated mathematical constants: 0, 1, π, e and i! But is it surprising?

Well, it needs a definition of $e^{i\pi}$, and it clearly follows from $e^{ix} = \cos x + i \sin x$ when $x = \pi$. How mysterious is the latter? To answer this question, we have to assign meaning to powers with complex exponents. Can this be done similarly to reals, where $e^x = \sum_{n=0}^{\infty}(x^n/n!)$? For complex z, we can try to define e^z using same power series $e^z := \sum_{n=0}^{\infty}(z^n/n!)$. Its convergence for every complex number z, and the property $e^{z_1}e^{z_2} = e^{z_1+z_2}$ for any complex z_1, z_2, is easy to establish (try the latter). Substituting $z = ix$ for real x, and combining real and imaginary parts (even without justification), gives $e^{ix} = \cos x + i \sin x$.

It is interesting that the formula $e^{ix} = \cos x + i \sin x$ was known before Euler. Roger Cotes published it 34 years before, describing the equivalent logarithmic relation in words.

Surprise 12. 100 women board an airplane with 100 seats. Each of them has a seat assigned. For some reason, the first woman who gets in takes a seat at random. Then the second passenger takes her own seat if it is not occupied (by the first), and picks a seat at random if her own seat is occupied. Then the third passenger takes her own seat if it is not occupied (by the first or second), and picks a seat at random if her own seat is occupied. And so on. What is the probability that the last person will sit in her own seat?

COMMENTARY. The answer is 1/2 (!), and it is not hard to prove it. I did it by constructing a recurrence for the sequence $\{p_n\}$, $n = 1, \ldots, 100$, where p_i is the probability that the ith woman sits in her own seat. A solution without any computations can be found in P. Winkler [**29**]. It was pointed out to the author by the editor that the problem generalizes; it works for 100 men, too.

Surprise 13. There exists a number of the form 111 ... 111 that is divisible by 2013.

COMMENTARY. This result is very surprising! It is also surprising that if the digit 1 in the desired number is replaced by any sequence of digits (e.g., 1776), and 2013 is replaced by any odd integer not divisible by 5, the result will still hold.

A solution below is an impressive application of the Pigeonhole Principle. Let $a_1 = 1$, $a_2 = 11$, $a_3 = 111$, and so on, $a_{2013} = 111\ldots111$ (2013 ones). Divide each a_n by 2013. If one number is divisible (the remainder is 0), then we are done. If not, two of the remainders must necessarily repeat, as there are at most 2012 distinct nonzero remainders. Subtracting the corresponding numbers, we obtain a number M that is divisible by 2013, and is of the form $111\ldots1111000\ldots000$. Hence $M = N \cdot 10^a$, where the digits of N are all 1's, and a is the number of zeros in M. Since M is divisible by 2013, and $\gcd(2013, 10^a) = 1$, then 2013 divides N.

Surprise 14. A person writes two distinct integers on two cards, one per card, and puts them on the table face down. Pick either of the two, look at it, and then guess whether the other number is larger or smaller. Suppose that you have a good random number generator. Prove that you have a strategy to make a correct guess with probability strictly greater than $1/2$.

COMMENTARY. The first time I heard this question, and its solution, was from Peter Winkler, at a dinner following his talk at the University of Pennsylvania many years ago. Though the proof was short and convincing, I have difficulties believing the statement. So does everyone to whom I tell this problem.

For a discussion and a solution, see D. Gale [**17**], where the problem is attributed to David Blackwell's modification of a related question.

Even after I see a proof, the fact is still mind boggling. I could not discover this.

These depend on how comfortable I am living with mysteries, and on being secure and honest with myself.

Surprise 15. Consider a continuous curve $y = f(x)$ on $[0, 1]$ such that $f(0) = f(1)$. A segment joining two points on the graph of the curve is called a chord. Consider only horizontal chords, i.e., those which are parallel to the x-axis. What lengths can they have?

COMMENTARY. The answer is very striking. It turns out that for any positive integer n, the curve will have a horizontal chord of length $1/n$, and that no other horizontal chord length is guaranteed! The last statement can also be phrased this way: For every α which is not a reciprocal of a positive integer, there exists a curve $y = f(x)$ that satisfies the conditions of the statement and that has no horizontal chord of length α.

A solution can be found in R. P. Boas [**7**], or in A. M. Yaglom and I. M. Yaglom [**30**]. See also comments in [**7**], concerning the history and applications of this problem.

Surprise 16. $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

COMMENTARY. As we know, finding the closed form $\pi^2/6$ for the sum of the series on the left (Basel problem), made young Euler a superstar. Here is a sketch of Euler's proof as presented in W. Dunham [**14**]:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} \quad \Rightarrow$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n} \quad \Rightarrow$$

$$\frac{\sin \sqrt{x}}{\sqrt{x}} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{n}.$$

Given a polynomial $a_0 + a_1x + \cdots + a_nx^n$ with nonzero roots $x_1, \ldots, x_n$, we have, by Viète's theorem,

$$\sum_{i=1}^{n} \frac{1}{x_i} = \frac{\sum_{i=1}^{n} x_1 \cdot \cdots \cdot x_{i-1}x_{i+1} \cdot \cdots \cdot x_n}{x_1x_2 \cdots x_n} = -\frac{a_1/a_n}{a_0/a_n} = -\frac{a_1}{a_0}.$$

Looking at $(\sin\sqrt{x})/\sqrt{x}$ as a polynomial of infinite degree (so what?), realizing that its roots are $x_n = \pi^2n^2$, $n \geq 1$, and applying Viète's theorem the same way (still holds, of course...), we get

$$\sum_{i=1}^{\infty} \frac{1}{\pi^2n^2} = -\frac{a_1}{a_0} = -\frac{-1/3!}{1} = \frac{1}{6}.$$

I would never be able to think of this! For many other proofs, see a recent article by D. Daners [**13**] and many references therein.

Surprise 17. Let $0 < r \leq R$, and $S(r, R) = \{x \in \mathbb{R}^3 : r \leq \|x\| \leq R\}$ be a uniform density spherical layer. Let A be any point inside it or on its inner surface. Then the gravity at A is zero.

COMMENTARY. I think that the result is impossible "to feel." If the Law of Gravity had 2 ± 10^{-100} as the exponent in the denominator, this would not be true. Still, Newton had an intuitive geometric argument with infinitesimals. It is described, e.g., by V. I. Arnold [**2**], or (an electrostatic version) in [**15**]. I did not find the argument convincing. The way I convinced myself that the fact was true was by using spherical coordinates, triple integrals, and Maple. I did not see this problem in calculus texts. Nor did I see problems asking to demonstrate that solid balls can be replaced by point masses at their centers, when we study motions of planets. I think these are great classical applications of triple integrals, and they should find a place in our courses; see, for example, [**26**].

The question was mentioned to me by Yves Crama, while we were driving on I-295 to the University of Delaware in 1988. We tried to find a simple explanation for it for several days, but could not. Do similar statements hold for annuli in one and two dimensions?

Surprise 18. Alice and Bob have one of two consecutive positive integers n and $n + 1$ written on their foreheads. Alice sees Bob's number, and Bob sees Alice's number. They alternate in asking another the same question: "Do you know your number?" Suppose Alice and Bob are infinitely intelligent: If there is a way to find out the number on their own forehead, then they will do it. Each of them can answer only "Yes," or "No." Prove that after finitely many question and answers, one of them will know their number.

COMMENTARY. The first time I heard this question was from Peter Winkler, at the same dinner I mentioned above. Can you prove it by using mathematical induction? See D. Gale [**17**].

Surprise 19. An automorphism f of a field is a bijection on it such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$. There are infinitely many automorphisms of the field of complex numbers $\mathbb{C}$.

COMMENTARY. The statement contrasts with the widely known facts that the only automorphism of the field of rational numbers or of the field of real numbers is the identity map.

I remember that this property of $\mathbb{C}$ was mentioned by L. A. Kaluźnin, in one of his lectures in early 1970's. Though the result was established at the beginning of the 20th century (see an interesting article by H. Kestelman [**22**]), it is still not known by many. It is easy to argue that if we ask for only continuous automorphisms of $\mathbb{C}$, then it can be either identity or conjugation. Two other surprising results about isomorphisms of algebraic structures are the following. The additive groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic, as are the multiplicative groups $\mathbb{C} \setminus \{0\}$ and $\{z \in \mathbb{C} : |z| = 1\}$. These two isomorphisms can be derived, for example, from the structure theorem of divisible abelian groups; see, e.g., L. Fuchs [**16**] or I. Kaplanski [**20**]. Maybe all these results become a bit less surprising if we note that the axiom of choice is used to establish them.

How is it possible that this problem is not solved yet?

These surprises are numerous. Listing a few of them, I tried to avoid famous unsolved problems (with few exceptions), and those problems I never thought about myself. I also picked ones that can be understood easily by most readers.

Surprise 20. What is the smallest number of people in a group such that there must be five of them who know one another or five who do not know one another?

COMMENTARY. This is a famous problem. The best-known result is that this number N satisfies $43 \leq N \leq 49$. If 5 in the statement of the problem is replaced by 2, the answer is 2. If 5 is replaced by 3, the answer is 6. If it is replaced by 4, the answer is 18. For details and related questions, see S. Radziszowski [**27**].

Surprise 21. Are there infinitely many positive integers n such that $\tan n > n$?

COMMENTARY. The question was asked by David Bellamy. It is instructive to experiment with Maple, and see that the positive integer solutions of this inequality are very rare. It can be shown that each of the inequalities $\tan n < -n$, and $\tan n > n/4$, have infinitely many solutions in positive integers, but the original problem is still open. See D. L. Bellamy, J. C. Lagarias, and F. Lazebnik [**4**].

Surprise 22. How many distinct points of intersection can n lines in a plane have? How many regions can they form?

COMMENTARY. The second question was asked by the author in 1998. It is easy to find the minimum and the maximum of these numbers. For the number of intersection points, it is 0 and $\binom{n}{2}$, respectively. For the number of regions, it is $n + 1$ and $\binom{n}{2} + \binom{n}{1} + \binom{n}{0}$, respectively. On the other hand, it is not clear which numbers can appear in between. For details and related results, see B. Grünbaum [**18**] and O. A. Ivanov [**19**].

Surprise 23. Let p be a prime, and $p \geq 5$. Take an arbitrary invertible $n \times n$ matrix A with entries in $\mathbb{Z}_p$ (the field of p elements), $n \geq 3$. It is conjectured that there always exists a vector $x = (x_1, x_2, \ldots, x_n)$ with all $x_i \in \mathbb{Z}_p$ such that no x_i is zero, and no component of xA is zero.

COMMENTARY. The statement is trivial over infinite fields. N. Alon and M. Tarsi [**1**] proved that the conjecture is true if $\mathbb{Z}_p$ is replaced by any finite field with a *nonprime* number of elements, more precisely, by $GF(q)$, where $q = p^e \geq 4$, p is prime, and

$e > 1$. For prime $q = p \geq 5$, and n much larger than p, the conjecture is still open. For some related results, see R. D. Baker, J. Bonin, F. Lazebnik, and E. Shustin [**3**], and Y. Yu [**31**].

I will stop here. Dear reader, please share with me your surprises.

Acknowledgment This article developed from the notes of a lecture given by the author in April 2007 to mathematics graduate students at the University of Delaware. The author is grateful to Jessica Belden, Brian Kronenthal, Xiaozhen Wen, and the referees for their numerous useful comments on the previous versions of this paper. This work was partially supported by NSF grant DMS-1106938.

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Summary In this article the author presents twenty-three mathematical statements that he finds surprising. Understanding most of the statements requires very modest mathematical background. The reasons why he finds them surprising are analyzed.

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Corrections

Frank Sandomierski's article [**1**] in our October, 2013 issue needs two corrections. Both apply to the section called "Simpson's rule" on page 263. The first equation in that section should be

$$E''(h) = \frac{1}{3}\left(g'(a+h) - g'(a-h)\right) - \frac{h}{3}\left(g''(a+h) + g''(a-h)\right).$$

The equation in the fourth line from the bottom of the page should be $h = (d-c)/2$.

We regret the errors, which were introduced in editing.

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NOTES

A Solution to the Basel Problem that Uses Euclid's Inscribed Angle Theorem

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We present a short, rigorous solution to the Basel Problem in the form

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Proof. Mercator's Formula

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

converges uniformly on the closed unit disk minus a neighborhood of -1. To see this, multiply by $1+z$ and note that the resulting series converges uniformly on the entire closed unit disk. In particular,

$$\log\left(1+e^{ix}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{inx} \quad \text{for } -\pi < x < \pi, \tag{1}$$

convergence being uniform on closed subintervals. We may therefore integrate termwise to obtain the antiderivative

$$F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in^2} e^{inx}.$$

This series even converges uniformly on the entire real axis. Continuity at $x = \pi$ thus gives the (improper) integral

$$\int_0^{\pi} \log\left(1+e^{ix}\right) dx = F(\pi) - F(0) = 2i \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

The imaginary part of the integrand is

$$\operatorname{Im} \log\left(1+e^{ix}\right) = \arg\left(1+e^{ix}\right) = \frac{x}{2},$$

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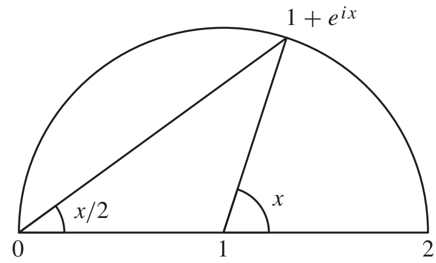

Figure 1 Applying the Inscribed Angle Theorem

where the second equality follows from the Inscribed Angle Theorem:

> *In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.*

This is Proposition 20 in Book III of Euclid's *Elements*, quoted here from Heath's translation [**2**]; see also FIGURE 1. Hence,

$$\operatorname{Im} \int_0^{\pi} \log\left(1+e^{ix}\right) dx = \frac{\pi^2}{4},$$

and the claim follows. ■

According to Heath [**3**, pp. 201–202], the main propositions of Book III of the *Elements* were known already to Hippocrates of Chios, a contemporary of Socrates.

Relation to other proofs The Basel Problem goes back to the middle of the seventeenth century. In its original form, it asks for the evaluation of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

As the terms with odd denominators make up three-fourths of the entire sum, it suffices to evaluate the sum of these terms, as we have done here. The reader is referred to Ayoub [**1**] for much interesting information on the history of the Basel Problem.

Euler gave four solutions to the Basel Problem, using Taylor series and product formulas for various trigonometric functions. Myriads of other solutions have been found since then, involving techniques from many different areas of mathematics. For an overview, the reader is encouraged to see the highly readable paper of Kalman [**4**]. We now discuss the relations between three previous solutions and the one given here.

Russell [**7**] essentially integrates the right-hand side of the *real* part

$$\operatorname{Re} \log\left(1+e^{ix}\right) = \log\left|1+e^{ix}\right| = \log\left(2\cos\frac{x}{2}\right).$$

Note, however, that, contrary to this proof, we have avoided appealing to Abel's Limit Theorem—which would in any case have been unjustified, since we are in effect integrating along the circle of convergence of the Mercator Series.

The reader might have noticed the Fourier series

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \quad \text{for } -\pi < x < \pi$$

lurking in the background. In fact, this classical formula follows directly from (1) by taking imaginary parts. A standard textbook proof now proceeds by invoking Parseval's Identity.

In two letters to Johann Bernoulli dated 6 and 9 November 1696, Leibniz discussed the Basel Problem and showed

$$\int \frac{\log(1+x)}{x} dx = \frac{x}{1} - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} \text{ etc.},$$

but his subsequent attack on the integral was unsuccessful; cf. Leibniz [**6**, pp. 174–180]. (Ayoub [**1**] quotes passages from the second of these letters but erroneously dates it to 1673.) Of course, the Basel Problem was to remain open until Euler's first solution in 1734. Ultimately, however, Leibniz was vindicated, as there *is* a direct way to evaluate the integral, using a suitable functional equation. The details and history of this proof are given by Kalman and McKinzie [**5**]. Alternatively, we can equip Leibniz's integral with the limits ± 1 and view the result as a contour integral along a straight line. If we then instead integrate along the semicircle $\gamma(x) = e^{ix}$, $0 \leq x \leq \pi$, thereby quite literally circumventing the roadblock mentioned in [**5**], we get the following elaboration of Leibniz's idea:

$$2\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \int_{-1}^{1} \frac{\log(1+x)}{x} dx = \operatorname{Im} \int_0^{\pi} \log\left(1 + e^{ix}\right) dx = \frac{\pi^2}{4}.$$

From a philosophical point of view, the idea of solving the Basel Problem by replacing an integral along the diameter of the unit circle with one along the perimeter explains the surprising appearance of π in a natural way. On the other hand, changing the path of integration obviously needs some justification, which can be avoided simply by integrating along the perimeter from the outset, as in the proof above.

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Summary We present a short, rigorous solution to the Basel Problem that uses Euclid's Inscribed Angle Theorem (Proposition 20 in Book III of the *Elements*) and can be seen as an elaboration of an idea of Leibniz communicated to Johann Bernoulli in 1696.

Characterizing Power Functions by Hypervolumes of Revolution

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In [**1**], Richmond and Richmond note that power functions—that is, functions of the form $f(x) = kx^\alpha$ for $k, \alpha > 0$—are characterized by a certain volume ratio associated with the surface of revolution generated by the graph of f.

Let $f : [0, \infty) \to [0, \infty)$ be a continuous function. Let $\mathcal{R}(r)$ be the first-quadrant region bounded by the curves $y = f(x)$, $x = r$, and the x-axis (FIGURE 1, left). Revolving $\mathcal{R}(r)$ about the y-axis yields a solid of revolution with volume $V(r)$. We will refer to the cylinder with the same base and height $f(r)$ as the *bounding cylinder* of this solid of revolution. Denoting by $C(r)$ the volume of the bounding cylinder, we define the *volume ratio* corresponding to f to be $V(r)/C(r)$ (FIGURE 1, right).

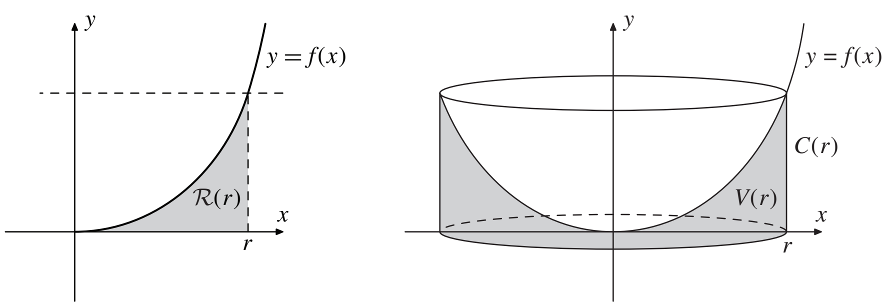

Figure 1 Volume of revolution

When f is a power function, $f(x) = kx^\alpha$, the volume ratio is given by

$$\frac{V(r)}{C(r)} = \frac{2}{2+\alpha}. \tag{1}$$

So, for example, the paraboloid of revolution ($\alpha = 2$) neatly breaks the volume of its circumscribing cylinder into two regions of equal volume. Note that the ratio in (1) is independent of r. According to [**1**], this occurs only when f is of the form $f(x) = kx^\alpha$. The power functions are exactly the functions that have constant volume ratios.

In this note, we extend the results of [**1**] to include n-dimensional ($n \geq 2$) hypersurfaces of revolution to find that power functions are similarly identified by an analogous ratio of hypervolumes.

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Higher dimensions

We define an n-dimensional *hypersurface of revolution* to be a hypersurface embedded in $\mathbb{R}^{n+1}$ such that the cross-sections orthogonal to a particular line, which we label the x-axis and call the *axis of revolution*, are $(n-1)$-dimensional spheres centered on the x-axis. The notation $S^{n-1}(f(x))$ refers to a cross-sectional $(n-1)$-dimensional sphere of radius $f(x)$, which bounds an n-dimensional ball, $B^n(f(x))$, and f is a nonnegative C^1 function which is called the *profile function* of the hypersurface of revolution. (See FIGURE 2, wherein the rotational axis is now the x-axis.)

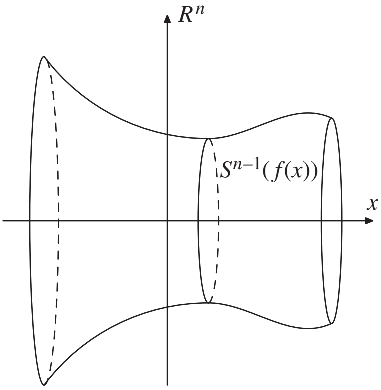

Figure 2 n-dimensional hypersurface of revolution in $\mathbb{R}^{n+1}$

The volume formulas for the sphere and ball are well known [**2**] and may be compactly written in terms of the *gamma function* (which is defined by the integral $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,\ x \geq 0$ and is sometimes called the *generalized factorial*, since for integer $n \geq 1$, $\Gamma(n) = (n-1)!$):

$$\mathrm{Vol}(B^n(r)) = \frac{2\pi^{n/2}}{n\Gamma(n/2)}r^n \quad \text{and} \quad \mathrm{Vol}(S^{n-1}(r)) = \frac{2\pi^{n/2}}{\Gamma(n/2)}r^{n-1}. \tag{2}$$

Note that the formulas in (2) follow the familiar pattern that $\frac{d}{dr}(\mathrm{Vol}(B^n(r)) = \mathrm{Vol}(S^{n-1}(r))$.

Referring once again to FIGURE 1 and using the formulas in (2), we can compute V and C and consequently their ratio. To compute V, we use the method of cylindrical shells wherein we take an infinitesimal slice of $\mathcal{R}$ parallel to the axis of revolution and then revolve this slice to produce a shell-shaped element, which we then slice open to form a rectangular slab. The n-volume of this elemental slab is $\mathrm{Vol}(S^{n-1}(x))f(x)dx$. The volume of the circumscribing hypercylinder C is easily computed using *Cavalieri's Principle* to give $\mathrm{Vol}(C(r)) = \mathrm{Vol}(B^n(r))f(r)$. Putting all this together yields

$$\frac{V(r)}{C(r)} = \frac{\int_0^r \mathrm{Vol}(S^{n-1}(x))f(x)\,dx}{\mathrm{Vol}(B^n(r))f(r)} = \frac{n\int_0^r x^{n-1}f(x)\,dx}{r^n f(r)}.$$

We are now in a position to develop our main result.

THEOREM. *Suppose that f is a positive, strictly increasing, twice differentiable function on an interval $(0, b)$. Then f is of the form $f(x) = kx^\alpha$ if and only if the* *hypervolume ratio satisfies*

$$\frac{V(r)}{C(r)} = \frac{n}{n+\alpha}$$

for all $r \in (0, b)$.

Proof. To prove that the constancy of the volume ratio corresponding to f implies that $f(x) = kx^\alpha$, we differentiate $V(r)/C(r)$ and set the result equal to 0 to obtain

$$[r^n f(r)][f(r)r^{n-1}] - (nr^{n-1}f(r) + r^n f'(r)) \int_0^r f(x)x^{n-1}\,dx = 0$$

or

$$\frac{r^{2n-1}(f(r))^2}{nr^{n-1}f(r) + r^n f'(r)} = \int_0^r f(x)x^{n-1}\,dx.$$

Another differentiation eliminates the integral from the above equation to give

$$\frac{(nf + rf')(nr^{n-1}f^2 + 2r^n ff') - (r^n f^2)(nf' + f' + rf'')}{(nf + rf')^2} = fr^{n-1}.$$

After a bit of algebraic manipulation all dependency on n disappears and we have

$$r(f')^2 - ff' - rff'' = 0.$$

This is exactly the equation obtained by Richmond and Richmond in [**1**], where they show that this implies $f(x) = kx^\alpha$.

To see necessity, let $f(x) = kx^\alpha$ to compute the hypervolume ratio

$$\frac{V(r)}{C(r)} = \frac{\int_0^r nx^{n-1}f(x)\,dx}{r^n f(r)} = \frac{n\int_0^r x^{n-1+\alpha}\,dx}{r^{n+\alpha}} = \frac{n}{n+\alpha}. \qquad \blacksquare$$

Geometrically, we can see why the ratio $V(r)/C(r)$ only depends on n and α by making the following change of scale, which does not affect the volume ratio between the revolved object and the cylinder. The hypersurface defined by the graph $y = kx^\alpha$, $x \in [0, r]$ is revolved around n other axes, which we label $y_1, \ldots, y_n$. Letting $u = x/r$ we can define a new coordinate system $v_1, \ldots, v_n$, where $v_i = y_i/kr^\alpha$. This transforms the picture to a graph of $v = u^\alpha$, drawn on the interval [0, 1] and revolved in the same way. Everything about the new graph is independent of k and r.

Acknowledgment The authors would like to thank Jeff Dodd for preliminary discussions.

REFERENCES

1. B. Richmond and T. Richmond, Characterizing power functions by volumes of revolution, *College Math. J.* **29** (1991) 40–41.
2. D. Smith and M. Vamanamurthy, How small is the unit ball? *Math. Mag.* **62** (1991) 101–107.

Summary A power function is characterized by a certain constant volume ratio associated with the surface of revolution generated by the graph of the function. We generalize this characterization to include hypersurfaces of revolution and find that power functions are similarly identified by the analogous ratio of hypervolumes of revolution. We write this ratio as an explicit function of the exponent of the power function and the dimension of the hypersurface.

A Pretzel for the Mind

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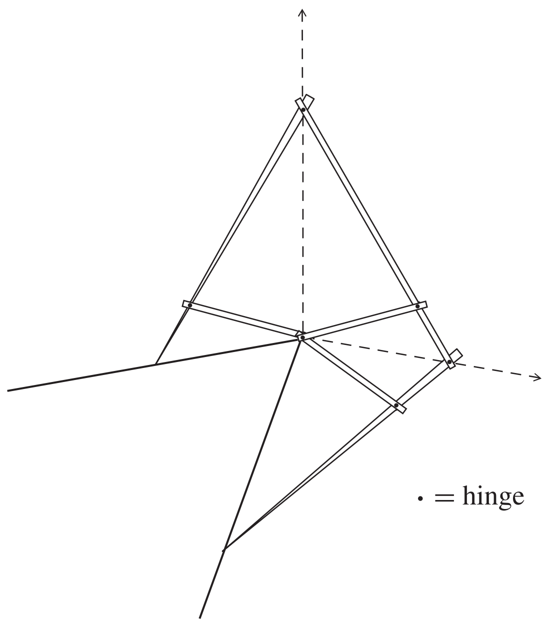
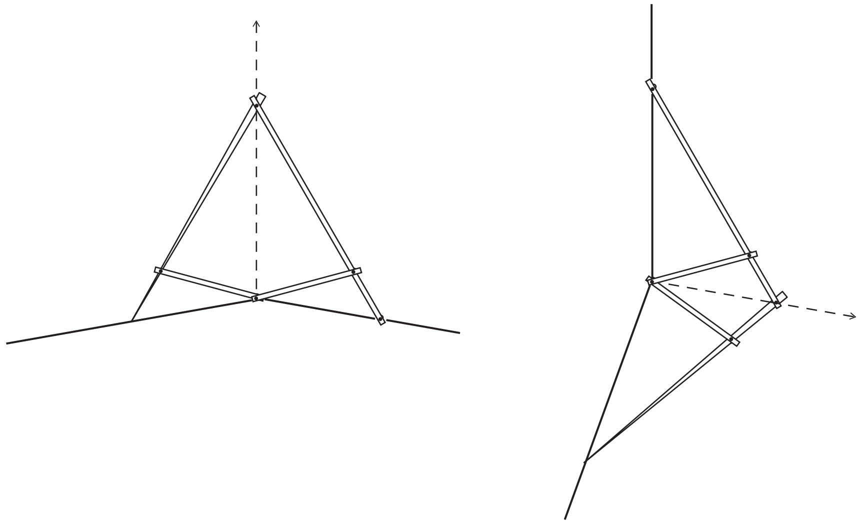
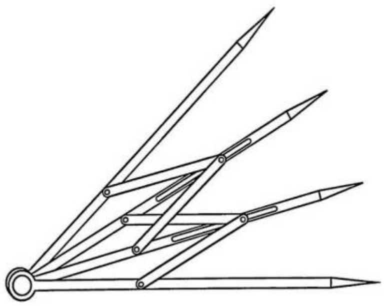


The mechanism shown above is a trisector: With its three-way hinge placed over the vertex and its free ends touching the rays of the angle to be trisected, the rays from the three-way hinge through the hinges between the long bars define the trisectors of the angle (here taken as 330 degrees). Conceptually, the device merges the two angle bisectors shown below.



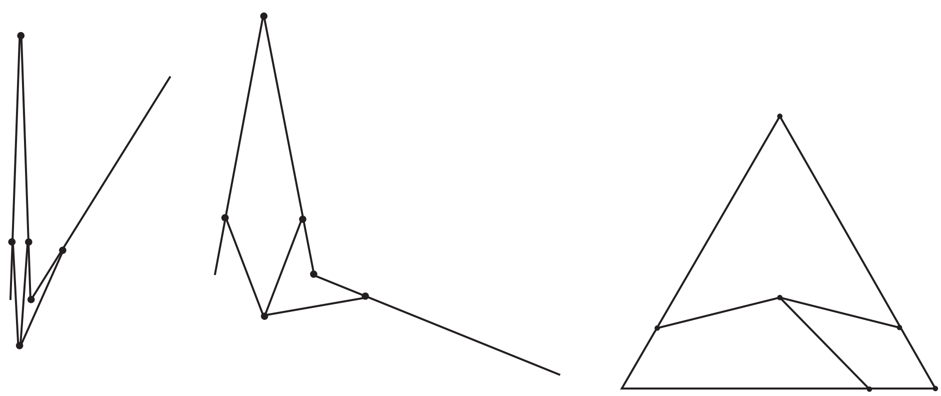
Math. Mag. **87** (2014) 228–229. doi:10.4169/math.mag.87.3.228.

It is a variant of the trisection compass shown below, proposed in 1875 by the French mathematician Charles-Ange Laisant, which is also made up of two bisectors so coupled as to move in lock-step.



Interested readers will find descriptions and illustrations of Laisant's mechanism and other mechanical trisectors in a review article by Robert C. Yates published in this MAGAZINE [**3**]. (The illustration above is from Isaacs [**1, 2**].)

The novel features of the current variant are the small number of parts and the absence of sliding contacts. It is shown below, in skeletal form, in the 30-degree and 150-degree configurations and the full pretzel shape at 360 degrees:



This trisector would be awkward to place on small angles, and indeed was never intended as a practical device. But it is constructible, with a few precautions. It is most easily designed in the 360-degree configuration. It then suffices to have the short bars link the center of the triangle to points well inside the outer thirds of the long bars. Also, the bars must be stacked so that each moves in its own plane, strictly back-to-front, rather than as shown (the drawing of Laisant's trisector is unrealistic in the same way, possibly because putting all the links in front makes the concept clearer).

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1. Gerald L. Alexanderson and Peter Ross, Twentieth-century gems from Mathematics Magazine, *Math. Mag.* **78** (2005) 110–123. http://dx.doi.org/10.2307/30044139
2. Rufus Isaacs, Two mathematical papers without words, *Math. Mag.* **48** (1975) 198.
3. Robert C. Yates, The trisection problem, *National Mathematics Magazine* **15(6)** (Mar. 1941) 278–293.

Summary A variant of Laisant's linkage-based trisector is described; its head-to-tail design has fewer parts and no slides.

PROBLEMS

PROPOSALS

To be considered for publication, solutions should be received by November 1, 2014.

1946. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Let $f : (0, \infty) \to (0, \infty)$ be a nondecreasing differentiable function. Characterize the functions $g : (0, \infty) \to (0, \infty)$ such that

$$2f(x)g(y) \leq f(x)g(x) + f(y)g(y),$$

for all $x, y \in (0, \infty)$.

1947. *Proposed by Raymond Mortini and Jérôme Noël, Université de Lorraine, Metz, France.*

Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} |\cos k| \geq \frac{n}{2}.$$

1948. *Proposed by Howard Carry Morris, Cordova, TN.*

For natural numbers m and n, it is known that $(mn)!$ is divisible by $(n!)^m$. Let

$$m = \max \{p^r : r \in \mathbb{N},\ p \text{ is prime, and } p^r \leq n\}.$$

Prove that $(mn)!$ is divisible by $(n!)^{m+1}$.

Math. Mag. **87** (2014) 230–237. doi:10.4169/math.mag.87.3.230. © Mathematical Association of America

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1949. *Proposed by Rob Downes, Mountain Lakes High School, Mountain Lakes, NJ.*

Let $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, and $\triangle A_3B_3C_3$ be triangles in the plane such that

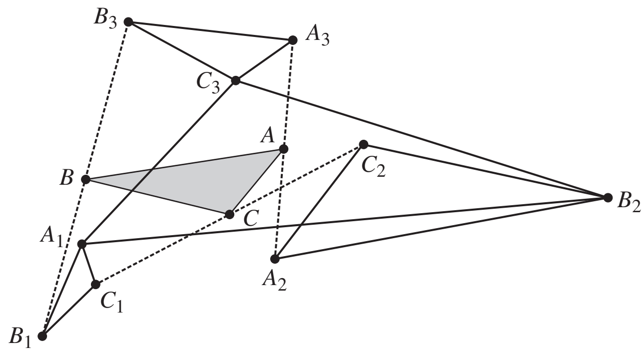
$$\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \sim \triangle A_3B_3C_3.$$

Let A be the midpoint of $\overline{A_2A_3}$, B the midpoint of $\overline{B_1B_3}$, and C the midpoint of $\overline{C_1C_2}$. Prove that

$$\triangle ABC \sim \triangle A_1B_1C_1 \text{ or } A = B = C$$

if and only if

$$\triangle A_1B_2C_3 \sim \triangle A_1B_1C_1 \text{ or } A_1 = B_2 = C_3.$$



1950. *Proposed by Aleksandar Ilic, Facebook, Inc., Menlo Park, CA.*

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\sum_{1 \leq i < j \leq n} a_i a_j (a_i + a_j)(2 - a_i - a_j) \leq \frac{8}{27},$$

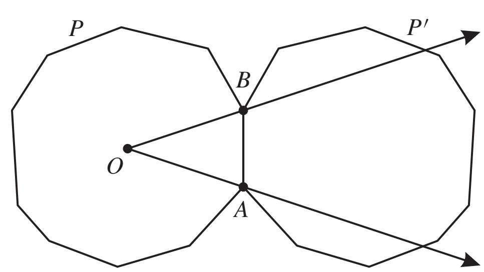
and determine when equality is achieved.

Quickies

Answers to the Quickies are on page 237.

Q1041. *Proposed by Seth Zimmerman, Evergreen Valley College, San Jose, CA.*

Let P be a regular polygon with $2n$ sides of length 1 and P' its reflection with respect to one of its sides $\overline{AB}$. Let O be the center of P. Characterize, as a function of n, the perimeter of the polygon obtained from the sides of P' that are entirely contained in the acute cone AOB. (The figure deliberately shows the construction for a nonregular polygon.)



Q1042. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let $n \geq 3$ be an integer, and let $x_1, x_2, \ldots, x_n$ be arbitrary integers. Show that

$$|x_1 - x_2| + |x_2 - x_3| + \cdots + |x_{n-1} - x_n| + |x_n - x_1|$$

is even.

Solutions

A functional inequality **June 2013**

1921. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.*

Let $f : (0, \infty) \to \mathbb{R}$ be a function such that

$$\frac{1}{2}\left(f(\sqrt{x}) + f(\sqrt{y})\right) = f\left(\sqrt{\frac{x+y}{2}}\right),$$

for every $x, y \in (0, \infty)$. Prove that

$$\frac{1}{n}\left(f(\sqrt{x_1}) + f(\sqrt{x_2}) + \cdots + f(\sqrt{x_n})\right) = f\left(\sqrt{\frac{x_1 + x_2 + \cdots + x_n}{n}}\right)$$

for every positive integer n and for every $x_1, x_2, \ldots, x_n \in (0, \infty)$.

I. *Solution by Shokhrukh Ibragimov (student), Lyceum Nr2 under the SamIES, Samarkand, Uzbekistan.*
For $x > 0$, let $g(x) = f(\sqrt{x})$. The given equation becomes

$$g(x) + g(y) = 2g\left(\frac{x+y}{2}\right) \text{ for } x, y \in (0, \infty). \tag{1}$$

Hence, for every $a, b, c, d \in (0, \infty)$ with $a + b = c + d$, we have

$$g(a) + g(b) = g(c) + g(d). \tag{2}$$

Now, plugging $x = t$ and $y = 2$ in (1), and $a = (t+2)/2$, $b = 1$, $c = t/2$, and $d = 2$ in (2), we obtain

$$g(t) = 2g\left(\frac{t}{2}\right) + k \text{ for } t \in (0, \infty), \tag{3}$$

where $k = g(2) - 2g(1)$. Now, plugging $t = t_1 + t_2$ in (3), and $x = t_1$ and $y = t_2$ in (1), implies that $g(t_1 + t_2) = g(t_1) + g(t_2) + k$ for all $t_1, t_2 \in (0, \infty)$. Recursively applying this equation yields

$$g(t_1 + t_2 + \cdots + t_n) = g(t_1) + g(t_2) + \cdots + g(t_n) + (n-1)\,k, \tag{4}$$

for all $n \in \mathbb{N}$ and $t_1, t_2, \ldots, t_n \in (0, \infty)$. Let $x_1, x_2, \ldots, x_n$ be given. Plugging first $t_i = x_i$ for $1 \leq i \leq n$, and then $t_1 = t_2 = \cdots = t_n = (x_1 + x_2 + \cdots x_n)/n$ in (4), we obtain

$$\begin{aligned}\frac{1}{n}\left(g(x_1)+g(x_2)+\cdots+g(x_n)\right) &= \frac{1}{n}\left(g(x_1+x_2+\cdots+x_n)-(n-1)\,k\right)\\ &= \frac{1}{n}\left(ng\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)+(n-1)\,k-(n-1)\,k\right)\\ &= g\left(\frac{x_1+x_2+\cdots+x_n}{n}\right),\end{aligned}$$

which implies the requested identity.

II. *Solution by Charles Martin, Vanderbilt University, Nashville, TN.*
For simplicity, let $g:(0,\infty)\to\mathbb{R}$ be defined by $g(t)=f(\sqrt{t})$. We proceed by a forward–backward induction on n, akin to Cauchy's proof of the Arithmetic Mean–Geometric Mean Inequality. First, we will prove the result for integers n of the form 2^k by induction on $k\geq 1$. The base case is simply the hypothesis of the theorem, so assume the result holds for some integer $n=2^k$. Given $x_1,\ldots,x_{2^{k+1}}\in(0,\infty)$, we can write

$$\begin{aligned}&[g(x_1)+\cdots+g(x_{2^k})]+[g(x_{2^k+1})+\cdots+g(x_{2^{k+1}})]\\ &\quad= 2^k\left[g\left(\frac{x_1+\cdots+x_{2^k}}{2^k}\right)+g\left(\frac{x_{2^k+1}+\cdots+x_{2^{k+1}}}{2^k}\right)\right]\\ &\quad= 2^{k+1}g\left(\frac{x_1+\cdots+x_{2^{k+1}}}{2^{k+1}}\right),\end{aligned}$$

which proves the claim for all n equal to a power of 2. Next, we prove that whenever the result holds for n, it also holds for $m<n$. Suppose that the m numbers $x_1,\ldots,x_m\in(0,\infty)$ have arithmetic mean A. Note then that the n numbers

$$x_1,x_2,\ldots,x_m,\underbrace{A,A,\ldots,A}_{n-m}$$

also have arithmetic mean A. Having assumed that the result holds for n numbers, we have

$$g(x_1)+\cdots+g(x_m)+\underbrace{g(A)+g(A)+\cdots+g(A)}_{n-m}=ng(A),$$

which rearranges into $g(x_1)+\cdots+g(x_m)=mg(A)$, as desired.

Also solved by George Apostolopoulos (Greece), Dionne Bailey, Robert Calcaterra, Elsie Campbell and Charles Diminnie, Michel Bataille (France), Rich Bauer, Yarema Boryshchak, Paul Budney, Hongwei Chen, Dana Clahane and Bill Cowieson, Marian Dincă (Romania), Bruce Ebanks, John N. Fitch, Eugene A. Herman, Omran Kouba (Syria), Eric Kouris (France), Victor Kutsenok, Harris Kwong, Elias Lampakis (Greece), Moti Levy (Israel), Reiner Martin (Germany), Peter McPolin (Northern Ireland), Northwestern University Math Problem Solving Group, ONU-Solve Problem Group, Sanjay K. Patel (India), Paolo Perfetti (Italy), C. G. Petalas (Greece), Ángel Plaza (Spain), Rudolf Rupp (Germany), Edward Schmeichel, Nicholas C. Singer, Skidmore College Problem Group, Texas State University Problem Solvers Group, Hongbiao Zeng, and the proposer. There were two incorrect solutions that used extra assumptions.

A medians and radii inequality **June 2013**

1922. *Proposed by Arkady Alt, San Jose, CA.*

Let m_a, m_b, and m_c be the lengths of the medians of a triangle with circumradius R and inradius r. Prove that

$$m_am_b+m_bm_c+m_cm_a\leq 5R^2+2Rr+3r^2.$$

Solution by George Apostolopoulus, Messolonghi, Greece.
Let E, F, and G be the midpoints of the sides AB, BC, and CA, respectively. We apply Ptolemy's Inequality to quadrilaterals $AEBD$, $BFEC$, and $CAFD$ to obtain

$$m_a m_b \le \frac{ab}{4} + \frac{c^2}{2}, \quad m_b m_c \le \frac{bc}{4} + \frac{a^2}{2}, \quad \text{and} \quad m_c m_a \le \frac{ca}{4} + \frac{b^2}{2}.$$

Adding up these inequalities, we get

$$m_a m_b + m_b m_c + m_c m_a \le \frac{a^2+b^2+c^2}{2} + \frac{ab+bc+ca}{4}.$$

By Heron's formula and $4srR = abc$ (equivalent to area $(ABC) = rs = \frac{1}{2}bc \sin A = abc/(4R)$), we get

$$a^2+b^2+c^2 = 2(s^2 - r^2 - 4rR) \quad \text{and} \quad ab+bc+ca = s^2 + r^2 + 4rR,$$

which means that

$$m_a m_b + m_b m_c + m_c m_a \le \frac{1}{4}(5s^2 - 3r^2 - 12rR).$$

Finally, by Gerretsen's Inequality $s^2 \le 4R^2 + 4rR + 3r^2$, we obtain the result

$$\begin{aligned} m_a m_b + m_b m_c + m_c m_a &\le \frac{5(4R^2 + 4rR + 3r^2) - 3r^2 - 12rR}{4} \\ &= 5R^2 + 2Rr + 3r^2. \end{aligned}$$

Equality holds when the triangle ABC is equilateral.

Also solved by Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Erhard Braune (Austria), Marian Dincă (Romania), Omran Kouba (Syria), Elias Lampakis (Greece), Moti Levy, (Israel), Peter Nüesch (Switzerland), Paolo Perfetti (Italy), and the proposer.

Looking for a closed-form expression **June 2013**

1923. *Proposed by Leonid Menikhes and Valery Karachik, South Ural State University, Chelyabinsk, Russia.*

Let m and n be nonnegative integers. Find a closed-form expression for the sum

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2m}{m-n+k}.$$

Solution by Ángel Plaza, Sergio Falcón, and José M. Pacheco, Universidad de Las Palmas de Gran Canaria, Spain.
Note that $\binom{2m}{m-n+k} = \binom{2m}{m+n-k}$. The proposed sum, $S_{n,m}$, may be written as

$$S_{n,m} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{2m}{m+n-k} = \frac{(2n)!\,(2m)!}{(n+m)!^2} \sum_{k=0}^{2n} (-1)^k \binom{n+m}{k} \binom{n+m}{2n-k}.$$

The last sum is the coefficient of x^{2n} in $(1-x)^{n+m}(1+x)^{n+m}$. Moreover, the coefficient of x^{2n} in $(1-x)^{n+m}(1+x)^{n+m} = (1-x^2)^{n+m}$ is also equal to $(-1)^n \binom{n+m}{n}$. Therefore,

$$S_{n,m} = \frac{(2n)!\,(2m)!}{(n+m)!^2} (-1)^n \binom{n+m}{n} = (-1)^n \frac{(2n)!\,(2m)!}{n!\,m!\,(n+m)!} = (-1)^n \frac{\binom{2n}{n}\binom{2m}{m}}{\binom{n+m}{n}}.$$

Editor's Note. As stated in some solutions, this result can be found in http://www.math.wvu.edu/~gould/Vol.4.PDF (use $x = m - n$ in Equation 7.26, p. 37).

Also solved by Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Robert Calcaterra, Joel Iiams, Omran Kouba (Syria), Harris Kwong, Elias Lampakis (Greece), Moti Levy (Israel), Nicholas C. Singer, and the proposer. There was one incorrect submission.

Adding a linear combination of harmonic series **June 2013**

1924. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Find a necessary and sufficient condition on (a_1, a_2, a_3, a_4) for the series

$$\sum_{n=0}^{\infty}\left(\frac{a_1}{4n+1}+\frac{a_2}{4n+2}+\frac{a_3}{4n+3}+\frac{a_4}{4n+4}\right)$$

to converge, and determine the sum of this series when that condition is satisfied.

Solution by Robert Calcaterra, University of Wisconsin–Platteville, Platteville, WI.
If the four terms of the general term of the given series are combined into a single rational function, the numerator will be cubic with lead coefficient $64(a_1 + a_2 + a_3 + a_4)$ and the denominator will be quartic with lead coefficient 64. Hence, the limit comparison test implies that the given series diverges if $a_1 + a_2 + a_3 + a_4 \neq 0$ by comparison to $\sum_{n=1}^{\infty} 1/n$, and converges if $a_1 + a_2 + a_3 + a_4 = 0$ by comparison to $\sum_{n=1}^{\infty} 1/n^2$. Therefore, the series converges if and only if $a_4 = -(a_1 + a_2 + a_3)$. Observe that if $a_1 = a_3 = 1$ and $a_2 = a_4 = -1$, then the given series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

which is known to converge to $\ln(2)$. Moreover, if $a_1 = a_2 = a_3 = 1$, $a_4 = -3$, and H_n denotes the sum of the first n terms of the harmonic series, then the partial sum from 0 to k of the given series equals

$$H_{4k+4} - H_{k+1} = (H_{4k+4} - \ln(4k+4)) - (H_{k+1} - \ln(k+1)) + \ln(4),$$

which converges to $\gamma - \gamma + \ln(4) = \ln(4)$. Here, γ denotes the Euler–Mascheroni constant. Lastly, let $a_1 = 1$, $a_2 = a_4 = 0$, and $a_3 = -1$. Then the given series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

which is known to converge to $\pi/4$. Notice that if $a_4 = -(a_1 + a_2 + a_3)$, then $4(a_1, a_2, a_3, a_4)$ may be written in the form $(a_1 - 2a_2 + a_3)(1, -1, 1, -1) + (a_1 + 2a_2 + a_3)(1, 1, 1, -3) + (2a_1 - 2a_3)(1, 0, -1, 0)$. It follows that the given series converges to

$$\frac{1}{4}\left[(a_1 - 2a_2 + a_3)\ln 2 + (a_1 + 2a_2 + a_3)\ln 4 + (2a_1 - 2a_3)\frac{\pi}{4}\right]$$
$$= (3a_1 + 2a_2 + 3a_3)\frac{\ln(2)}{4} + (a_1 - a_3)\frac{\pi}{8}.$$

Also solved by George Apostolopoulos (Greece), Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Hongwei Chen, Bill Cowieson, Chip Curtis, John N. Fitch, Dmitry Fleischman, Ovidiu Furdui (Romania), Eugene A. Herman, Victor Kutsenok, Elias Lampakis (Greece), Moti Levy (Israel), James Magliano, Hosam M. Mahmoud, Reiner Martin (Germany), Paolo Perfetti (Italy), Ángel Plaza (Spain), Rudolf Rupp (Germany), Nicholas C. Singer, John H. Smith, Stan Wagon, John Zacharias, and the proposer. There were two incomplete submissions.

Finding when the inverse and the sum commute **June 2013**

1925. *Proposed by Tim Kröger and Rudolf Rupp, Georg Simon Ohm University of Applied Sciences, Nürnberg, Germany.*

Probably every mathematician teaching undergraduate mathematics has experienced the difficulty of persuading students that the equation $(A+B)^{-1} = A^{-1} + B^{-1}$ is not true for arbitrary matrices A and B. However, the equation is true for *some* matrices A and B.

For every positive integer n, determine all pairs of $n \times n$ real matrices A and B such that $(A+B)^{-1} = A^{-1} + B^{-1}$.

Solution by Michel Bataille, Rouen, France.
Let A and B be invertible $n \times n$ real matrices. First, we show that $(A+B)^{-1} = A^{-1} + B^{-1}$ if and only if the matrix $C = B^{-1}A$ satisfies $C^2 + C + I_n = 0$, where I_n is the unit $n \times n$ matrix.

If $(A+B)^{-1} = A^{-1} + B^{-1}$, then $I_n = (A^{-1} + B^{-1})(A+B)$, hence

$$A^{-1}B + B^{-1}A + I_n = 0. \tag{1}$$

It follows that $BA^{-1}B + A + B = 0 = B + AB^{-1}A + A$ and so $BA^{-1}B = AB^{-1}A$, that is, $A^{-1}B = (B^{-1}A)^2$, and back to (1), we obtain $C^2 + C + I_n = 0$. Conversely, if $C = B^{-1}A$ satisfies $C^2 + C + I_n = 0$, then we have $C(C + I_n) = -I_n$; hence, $A^{-1}B = C^{-1} = -C - I_n = -B^{-1}A - I_n$ and $I_n = (A^{-1} + B^{-1})(A+B)$ follows, proving that $A+B$ is invertible with $(A+B)^{-1} = A^{-1} + B^{-1}$.

It remains to identify the $n \times n$ real matrices C such that $C^2 + C + I_n = 0$. Let C be such a matrix. Since the polynomial $x^2 + x + 1$ (clearly the minimum polynomial of C) has only simple roots, namely $\omega = e^{2\pi i/3}$ and $\overline{\omega} = e^{-2\pi i/3}$, C is diagonalizable in $\mathcal{M}_n(\mathbb{C})$ and its only possible eigenvalues are ω and $\overline{\omega}$. The characteristic polynomial of C is of the form $\chi(x) = (x-\omega)^k (x - \overline{\omega})^{n-k}$ for some integer k such that $0 \le k \le n$. Observing that this polynomial is in $\mathbb{R}[x]$, we must have $k = n-k$, hence $n = 2k$ and $\chi(x) = (x^2 + x + 1)^{n/2}$. If $n = 2$, this means that $C = KC_2K^{-1}$ where

$$C_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

is the companion matrix of $x^2 + x + 1$ and K is an arbitrary matrix in $GL_2(\mathbb{R})$. If $n > 2$, the general form of the solutions C is $K\operatorname{diag}(C_2, C_2, \ldots, C_2)K^{-1}$, where $K \in GL_n(\mathbb{R})$ and the notation $\operatorname{diag}(C_2, C_2, \ldots, C_2)$ represents the diagonal sum of $n/2$ matrices C_2. Conversely, any matrix of this form satisfies $C^2 + C + I_n = 0$, since $C_2^2 + C_2 + I_2 = 0$.

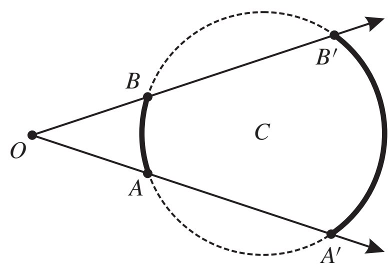
In conclusion, for even n the required pairs are given by $(A, B) = (BC, B)$ where $B \in GL_n(\mathbb{R})$ and C is of the form described above; for n odd, there are no suitable pairs.

Also solved by Daniel López Aguayo (México), George Apostolopoulos (Greece), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Robert Calcaterra, Bill Cowieson, Michael Goldenberg and Mark Kaplan, Jason A. Green, Eugene A. Herman, Roger A. Horn, Dan Jurca, Omran Kouba (Syria), Elias Lampakis (Greece), Moti Levy (Israel), Charles Martin, Peter McPolin (Northern Ireland), Nicholas C. Singer, Jeffrey Stuart, Nora Thornber, Hongbiao Zeng, and the proposers. There was one incorrect submission.

Answers

Solutions to the Quickies from page 231.

A1041. Regardless of n, there are always three sides of P' in the cone AOB, so the requested perimeter equals 3. Using an appropriate dilation, suppose that P' is circumscribed by a circle C of radius 1. Let $A' \neq A$ and $B' \neq B$ be the intersections of C with the rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively. The diagram is now that of a familiar Euclidean theorem. Because the angle at O opens to $\overline{AB}$, which is one of the $2n$ sides of P, it measures $2\pi/2n = \pi/n$. The arc AB similarly measures π/n. Because $\pi/n = \angle AOB = (\widehat{A'B'} - \widehat{AB})/2 = (\widehat{A'B'} - \pi/n)/2$, it follows that $\widehat{A'B'} = 3\pi/n$, which is the angle subtending three sides.



A1042. By considering the definition of absolute value and eliminating absolute value signs, in the sum each x appears exactly twice, either as x and x, or $-x$ and $-x$, or x and $-x$. In any case, the sum is an even integer.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Sun, Albert, To divide the rent, start with a triangle, *New York Times* (29 April 2014) D2, http://www.nytimes.com/2014/04/29/science/to-divide-the-rent-start-with-a-triangle.html.

Su, Francis Edward, Rental harmony: Sperner's lemma in fair division, *American Mathematical Monthly* 106 (1999) 930–942, http://www.math.hmc.edu/~su/papers.dir/rent.pdf.

Klarreich, Erica, Math shall set you free—from envy: How to do divorce, divestment, and death properly, http://nautil.us/issue/13/symmetry/math-shall-set-you-freefrom-envy.

The American public is skeptical about science, dubious of the value of mathematics beyond arithmetic, and suspicious of both scientists and mathematicians. In fact, the first commenter on the *Times* article above said he would never share an apartment with its author. Nevertheless, here is highly practical mathematics that many people can use, profit from, and enjoy—and it's nothing like the "math" they learned (to hate) in school. Sun's article about dividing rent among apartment mates comes at an ideal time for students (such as one of my sons) who will move in the fall; but it does not mention the equivalence of optimality of this procedure to various fixed-point theorems, nor the connection to the proof that the game of Hex must have a winner. For those mathematical associations, you need to hear one of Francis Su's memorable lectures, as he travels in his role as incoming president of the MAA. Klarreich points out the potential inconsistency of envy and "efficiency" in fair division. Anyway, how come, 15 years after Su's article, the mathematics is only now reaching (some of) the public, thanks to journalist/programmer Sun, recently graduated from college, rather than from a mathematician/expositor?

Pagano, Marcell, and Sarah Anoke, Mommy's baby, daddy's maybe: A closer look at regression to the mean, *Chance* 26 (3) (2013) 4–9.

Many mathematicians, particularly at small colleges, find themselves teaching statistics (perhaps in competition with the economists, the biologists, ...). A data set featured in many introductory textbooks in connection with regression to the mean is Galton's data on heights of sons and daughters vs. the heights of their parents. Galton found that the coefficient of the mother's height in the regression was higher than the father's, attributed that to women being shorter, and found it expedient to multiply the heights of the mothers by 1.08—thereby allowing for the mother's height to contribute more to the heights of the offspring. The authors of this article appear to be the first to suggest an alternative explanation, that 10%–15% of the fathers were not the biological fathers, and explore the plausibility of that in terms of the statistics involved.

Rehmeyer, Julie, Visions of math, *Discover* (April 2014) 32–37.

This article depicts works of art in various media (crocheted, digital, wood) that were inspired by mathematical structures and displayed at recent Joint Mathematics Meetings: a hyperbolic lamp shade, Brownian motion, fractals, Apollonian gaskets, decagrams (10-pointed stars), and tetrahedra. Full text and illustrations of the article is available online only to subscribers, but others can view other examples at http://DiscoverMagazine.com/MathArt.

Math. Mag. **87** (2014) 238–239. doi:10.4169/math.mag.87.3.238.

Rosenthal, Jeffrey S., Statistics and the Ontario Lottery retailer scandal, *Chance* 27 (1) (2014) 4–9.

Want to become famous (briefly)? Use statistics to expose thieves. In a notorious case, a store clerk deceived a customer about a winning lottery ticket and claimed the winnings herself; a judge eventually ordered the rightful owner to be paid by the lottery company, which insisted that he not disclose the payment. Statistician Rosenthal was asked by the Canadian Broadcasting Company how often such theft by clerks might be happening. Answer, from a simple data analysis: A lot. Absolutely surely. And everywhere: not just Ontario, but British Columbia, Nova Scotia, the Western Canada Lottery, Québec—and not to make Canada look bad, also Arizona, Texas, Iowa, California, Minnesota, ... Reforms in Ontario include no longer allowing clerks to purchase tickets at their own stores, and requiring winners to sign lottery tickets before redeeming them. Rosenthal cleaned up Canada; what's it like these days in your state?

Sfard, Anna, Why mathematics? what mathematics?, *Mathematics Educator* 22 (1) (2012) 3–16. Reprinted in *The Best Writing on Mathematics 2013*, edited by Mircea Pitici, 130–14; Princeton University Press, 2014.

Author Sfard (University of Haifa) reviews briefly, deconstructs, and dismisses most of the arguments in favor of the "prominence" of mathematics in school curricula. The utilitarian argument: People don't need much, can learn what they need on the job, and can't even apply what they do know. Mathematics "empowers": Power over whom, and why institute such hegemony? The cultural argument: Mathematics as a crucial element of one's identity—you are incomplete as a person without it? The perfect selection tool: A harmful practice! Instead, Sfard suggests a vision of mathematics as a form of discourse, for telling stories about the world. School mathematics should consequently be framed as the art of communicating ("ambiguity-proof," "saying more with less," "capacity for generalization") and as a basic literacy (which involves knowing when to switch into, or out of, mathematical discourse, and how to incorporate "metamathematical rules of communication into other discourses").

Wolchover, Natalie, To settle infinity dispute, a new law of logic, *Quanta Magazine: Illuminating Science* (26 November 2013) http://www.simonsfoundation.org/quanta/20131126-to-settle-infinity-question-a-new-law-of-logic/.

The Simons Foundation, founded by a former mathematics professor at SUNY–Stony Brook who became rich as a hedge-fund owner, sponsors an online magazine to enhance public understanding of science and mathematics. Articles about biology and physics predominate, but mathematics and computer science get a share. This article is slightly mistitled; what is meant is a new axiom for set theory. The leading candidates for additions to the traditional Zermelo-Fraenkel set theory with the axiom of choice (ZFC) are forcing axioms and the "inner-model" axiom that "V = ultimate L." ZFC cannot decide the continuum hypothesis (CH), which says that $\aleph_1 = 2^{\aleph_0}$: there is no infinity between that of the integers, $\aleph_0$, and that of the reals, $c = 2^{\aleph_0}$. Forcing forces the CH to be false, while asserting that the universe of sets is "ultimate L" makes it true. The key question is what the goal is, and what the effect on non-logician mathematicians might be. Some set-theorists favor a policy of "let 100 flowers bloom"—explore all universes of set theories. I suspect that most mathematicians would prefer, for elegance and simplicity, that $c = 2^{\aleph_0}$. Many just ignore the higher (and higher) infinities of set theory, knowing that such "monsters" have no relevance for their work and suspecting that they may not even "exist."

Bleicher, Ariel, The Oracle, *Scientific American* 310 (5) (May 2014) 70–75.

The title of this article appears above a well-known photo of Ramanujan; but he is not the Oracle. Nor is Ken Ono (Emory University), who is the real subject of the article, which discusses his research on the partition function $p(n)$ = the number of combinations of positive integers that sum to n (for example, $p(4) = 5$). Ramanujan proved the "partition congruences" that $5 \mid p(5k + 4)$ (5 divides $p(5k + 4)$), $7 \mid p(7k + 4)$, and $11 \mid p(11k + 4)$. But $13 \nmid p(13k + 4)$, though $17^3 \mid p(17^3k + 4)$ and $23^6 \mid p(23^6k + 4)$. What is the pattern? Ono figured it out; in fact, for every prime, there is a power for which there is a partition congruence. The Oracle? It is a calculator constructed by Ono and Jan Bruinier (Technical University of Darmstadt) to calculate large partition numbers—"the holy grail Ramanujan never obtained."

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